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THE LOCATION OF MAXIMUM PRINCIPAL STRESSES

T. Ranov[1] and H. S. Wolko[2]
(Proc. Paper 1629)

SUMMARY

While in a great number of practical beam applications the location of the numerically largest principal stresses and maximum shearing stresses presents no special problem, i.e. they occur in the critical sections at the outermost fibers and neutral axis respectively, it is well-known that cases exist in which these stress extrema are induced at intermediate fibers. It appears, however, from a reasonably thorough search of the literature, that this question has not been exhaustively treated and, so far, no general method or design criterion has been offered to provide a rapid check on the possible occurrence of the stress extrema at such intermediate locations. This paper summarizes such a method[3] and presents it in the form of design charts for two common cross-sectional shapes. In these two cases, it should be possible to ascertain with relative ease whether or not a given beam must be designed on the basis of these "shifted" maximum stresses or not. The treatment shows further that only computation of a few comparatively simple characteristic parameters is required for a solution of a given problem.

General Case

Let it be required to examine the stress distribution in an arbitrarily loaded beam of a cross-sectional shape and other properties which permit an exact (or approximate) treatment with reasonable adequacy by the elementary theory of elastic bending and shear in beams. Assume also that any tensile or compressive axial loading which may be present is linearly superposable on the bending stresses, i.e. eliminate any cases of impending instability due to compressive buckling.

Note: Discussion open until October 1, 1958. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. Paper 1629 is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 84, No. ST 3, May, 1958.

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2. Research Engr., Bell Aircraft Corp., Niagara Falls, N. Y.
3. Submitted in part by H. S. Wolko as a Master of Science thesis at the Univ. of Buffalo.

With these restrictions in mind, we first write down the general expressions for the principal stresses and maximum shearing stress at a typical point in a beam cross-section.

$$\sigma_{1,2} = \frac{\sigma_x}{2} \pm \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} \tag{1}$$

$$\tau_{max.} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} \tag{2}$$

Considering that, for a given cross-section, the elementary theory stress components

$$\sigma_x = \frac{My}{I} + \frac{P}{A} \tag{3}$$

and

$$\tau_{xy} = \frac{VQ}{Ib} \tag{4}$$

are both functions of the fiber distance y, the principal and maximum shearing stresses may be examined mathematically for the occurrence of maxima and minima. Differentiating (1) and (2) with respect to y, we obtain, after some algebraic transformation,

$$\frac{\sigma_x'^2}{\tau_{xy}'^2} - \frac{2\sigma_x}{\tau_{xy}} \frac{\sigma_x'}{\tau_{xy}'} - 4 = 0 \tag{5}$$

and

$$\frac{\sigma_x'}{\tau_{xy}'} + 4\frac{\tau_{xy}}{\sigma_x} = 0 \tag{6}$$

where

$$\sigma_x' = \frac{d\sigma_x}{dy} \tag{6a}$$

and

$$\tau_{xy}' = \frac{d\tau_{xy}}{dy} \tag{6b}$$

Inserting into these equations the stress components (3) and (4) and their derivatives, and simplifying, we have, in dimensionless form

$$\frac{C^2}{4}\left(\frac{k}{F'}\right)^2 = 1 + \frac{k}{2F'}\left(C^2\frac{yk}{F} + BC\frac{k^2}{F}\right) \tag{7}$$

and

$$C\left(\frac{k}{F'}\right) + \frac{4F}{Cyk + Bk^2} = 0 \qquad (8)$$

Here the following new symbols have been introduced, replacing the well-known standard notation thus far used:

$$C = \frac{M}{Vk} = \text{GENERAL "LOAD - SECTION" PARAMETER}$$

$$k = \text{RADIUS OF GYRATION OF SECTION ABOUT NEUTRAL AXIS}$$

$$F = \frac{Q}{b}$$

$$F' = \frac{dF}{dy}$$

$$B = \frac{P}{V} = \text{RATIO OF AXIAL LOAD TO LOCAL SHEARING FORCE}$$

Inspecting (7) and (8), and re-examining them as to their physical significance, we observe that for a specific cross-sectional shape they represent relationships between the parameters C and B on the one hand, and the fiber distance y on the other, derived from the condition for mathematical extrema of the principal and maximum shearing stress functions. If the above relationships between B, C and y can be explicitly set up for a given shape of beam cross-section in analytical or graphical form, their solutions will supply the necessary criteria for the existence and location of principal and maximum shearing stress extrema within a beam of this particular cross-section.

Rectangular Section

It appears at this stage from an inspection of (7) and (8) that no useful purpose can be served in continuing this investigation in a general form. As a first special case, therefore, a rectangular section is examined and the above-mentioned criteria ultimately developed for it in chart form. The characteristic quantities of a specific cross-section required for (7) and (8) are F and k. For a rectangle of width b and depth h they become

$$F = \frac{Q}{b} = \frac{1}{b}\int_{y}^{h/2} by\,dy = \frac{1}{2}\left[\left(\frac{h}{2}\right)^2 - y^2\right] \qquad (9)$$

and

$$k = \frac{h}{2\sqrt{3}}\ , \qquad (10)$$

both obviously independent of b.

In addition,

$$F' = \frac{dF}{dy} = -y . \tag{11}$$

For a more direct and immediate check, it is also convenient to replace the parameter C by the modified load-section parameter

$$C_h = \frac{M}{Vh} .$$

This is done with the aid of the easily obtained conversion

$$C = 2\sqrt{3}\, C_h .$$

Substitution of these quantities in (7) and (8) and introduction of the symbol Z for 2y/h yields, after simplification,

$$\frac{Z^4}{C_h} + \left(3C_h - \frac{1}{C_h}\right) Z^2 + C_h = -\frac{2}{3} BZ \tag{12}$$

and

$$\frac{Z^3}{C_h} + \left(2C_h - \frac{1}{C_h}\right) Z = -\frac{1}{3} B \tag{13}$$

for principal and maximum shearing stress extrema respectively. Remembering that for any given value of C_h the real roots of these equations for Z represent fiber distance parameters at which stress maxima or minima exist, we further realize that any values of Z larger than unity are outside the cross-section and therefore of no interest. The real roots of (12) and (13) are obtained graphically for different values of the parameters C_h and B, and are shown in Figs. 1 and 2 within the bounds of the cross-sectional depth $Z = \pm 1$. The limiting case of zero axial load ($B = 0$) is represented by the Z-axis. It is of interest to calculate the values of C_h which will just give real roots of (12) and (13). This is achieved by noting that, with $B = 0$, these equations assume the forms:

$$Z^4 + \left(3C_h^2 - 1\right) Z^2 + C_h^2 = 0 \tag{14}$$

$$Z^3 + \left(2C_h^2 - 1\right) Z = 0 \tag{15}$$

and have real roots only if

$$\left|C_h\right| \leq \frac{1}{3} \tag{16}$$

and

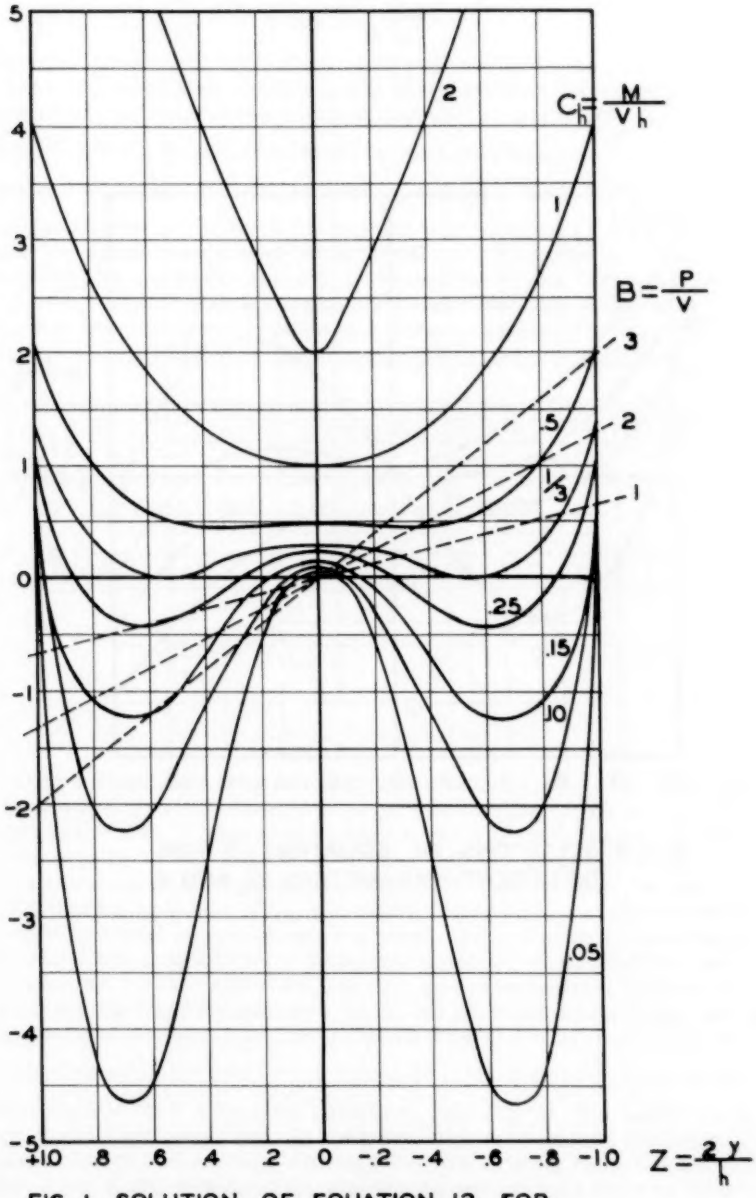

FIG. 1 SOLUTION OF EQUATION 12 FOR DIFFERENT PARAMETERS C_h AND B.

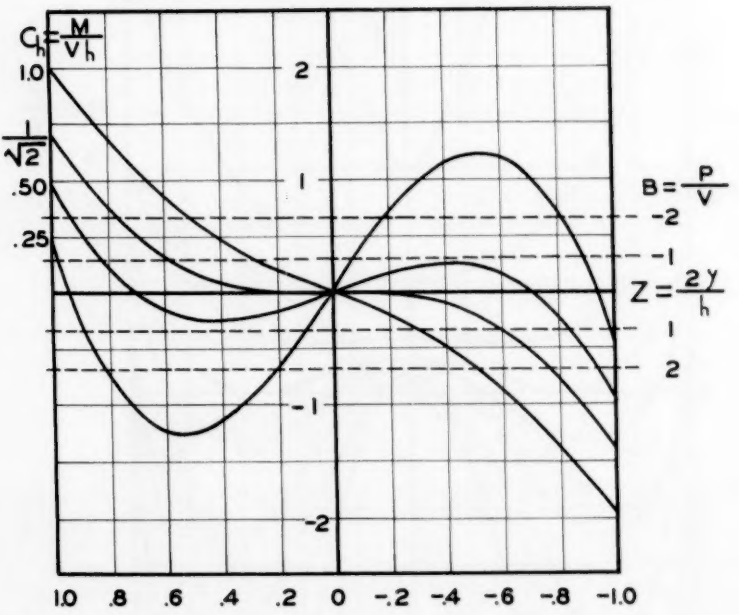

FIG. 2 SOLUTIONS OF EQUATION 13 FOR DIFFERENT PARAMETERS C_h AND B.

$$\left|C_h\right| \leq \frac{1}{\sqrt{2}} \tag{17}$$

respectively. These inequalities may be interpreted in words as meaning that, in the case of pure bending combined with shear, intermediate-fiber principal stress extrema will exist if the modified load-section parameter $C_h = M/Vh$ is between $-\frac{1}{3}$ and $+\frac{1}{3}$, while intermediate-fiber maximum shearing stress extrema will be present if C_h is between $-\frac{1}{\sqrt{2}}$ and $+\frac{1}{\sqrt{2}}$. A set of charts which permits the location of the fiber distance for which a given C_h will produce principal and maximum shearing stress extrema is presented in Figs. 3 and 4. These figures are applicable only to rectangular beams with purely transverse loading, but with entirely arbitrary support conditions. The only requirement is that it must be possible to determine analytically or graphically the bending moments and shearing forces for the cross-sections to be investigated.

For convenience of interpretation of the significance of the results obtained from Figs. 3 and 4, the principal and maximum shearing stress distributions are plotted for the case of the transversely loaded rectangular beam, with C_h as a parameter, in Figs. 5 and 6. In preparation for this, the equations (1) and (2) were first rewritten in the dimensionless forms:

$$\frac{\sigma_{1,2}}{3\tau_a} = \left[C_h Z \pm \sqrt{C_h^2 Z^2 + \frac{1}{4}\left(1 - Z^2\right)^2} \right] \tag{18}$$

$$\frac{\tau_{max.}}{3\tau_a} = \sqrt{C_h^2 Z^2 + \frac{1}{4}\left(1 - Z^2\right)^2} \tag{19}$$

where

$$a = \frac{V}{A} = \text{average shearing stress.}$$

Figs. 5 and 6 graphically illustrate the changes which the principal and maximum shearing stress distributions undergo with varying parameter C_h. The limiting case of zero bending, represented by the curve $C_h = 0$, clearly confirms the equality in this case of both principal and maximum shearing stresses to the value of the shearing stress on the neutral axis. On the other hand, for increasing values of C_h, i.e. rising contribution, for a given depth, of the bending stress compared with the shear, the curves tend to assume the shape usually associated with the presence of extreme-fiber maxima. Between these two limiting situations lies the class of stress distributions on which our attention is here centered, i.e. those for which maxima and minima (of the principal stress curves) move inward from the neutral axis and the outer fiber respectively until, for a "critical" value of $C_h (= \frac{1}{3})$, they merge into one single point of horizontal inflection. (See Fig. 5) The dashed lines in Figs. 5 and 6 are drawn through the points of maximum (and minimum) stresses of the plotted curves. Although they are directly related to the plots of Figs. 3 and 4, and thus could be used for the approximate determination of the intermediate-fiber extrema, the latter figures are recommended for accurate checks, particularly when C_h - values are given and Z - values are sought.

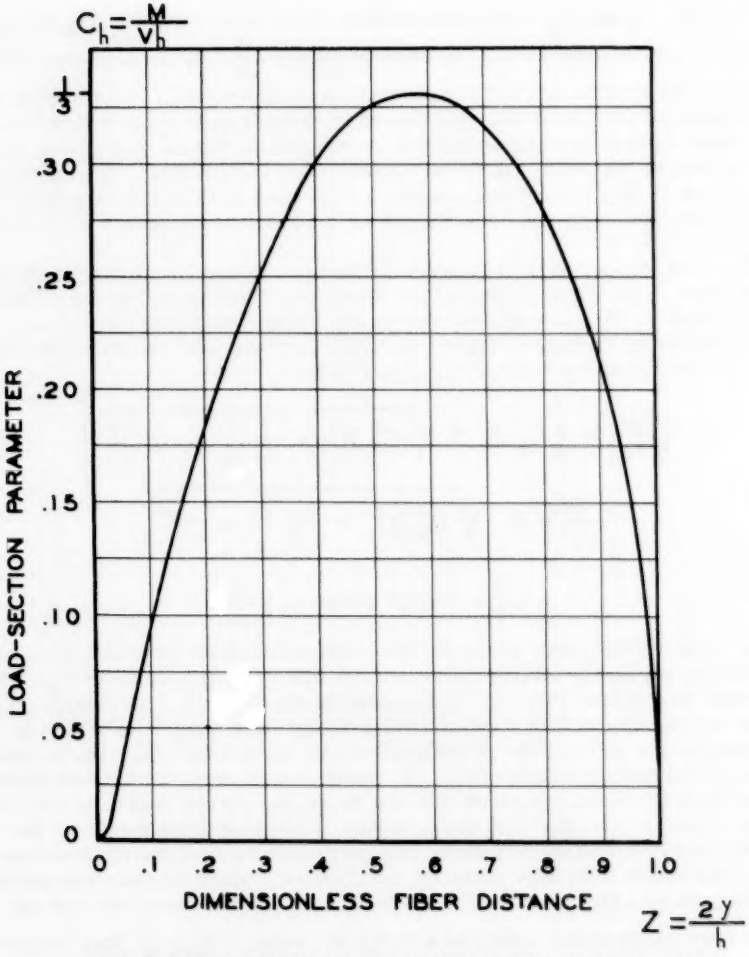

FIG.3 RECTANGULAR LOAD-SECTION PARAMETER VS. FIBER DISTANCE FOR PRINCIPAL STRESS EXTEMA.

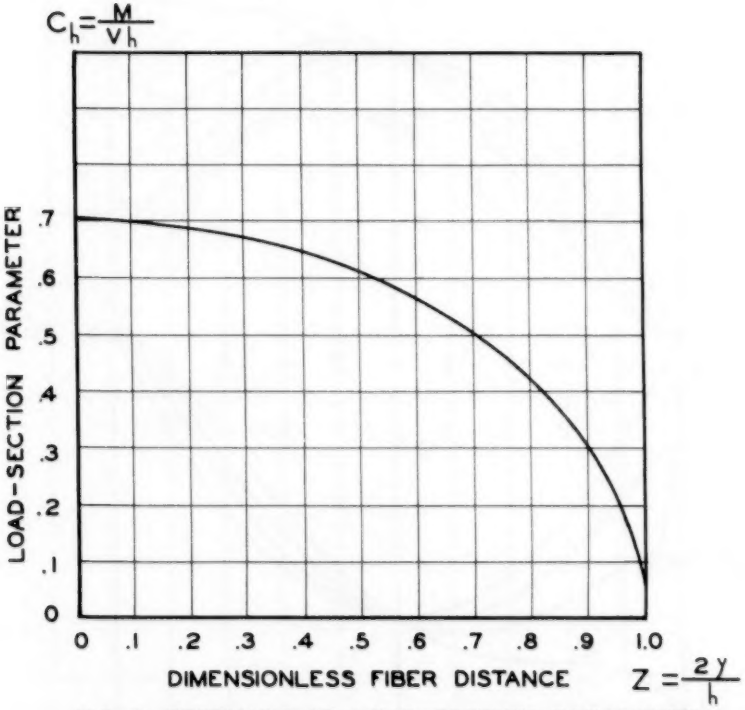

FIG. 4 RECTANGULAR LOAD-SECTION PARAMETER VS. FIBER DISTANCE FOR MAXIMUM SHEAR EXTREMA.

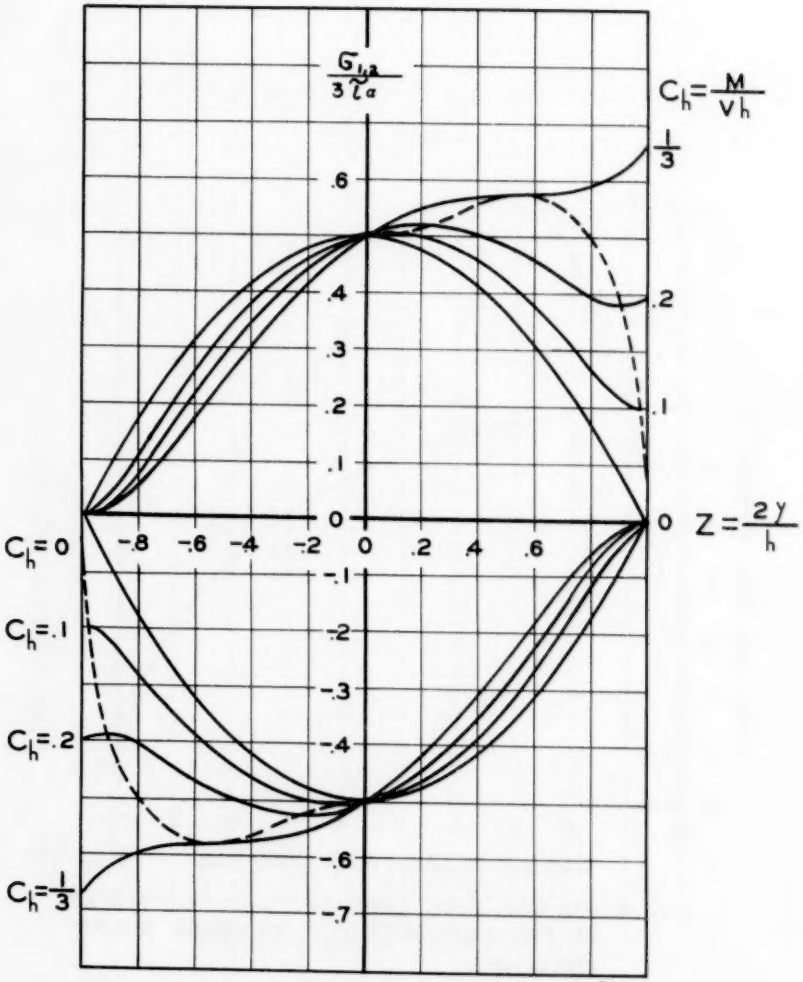

FIG. 5 PRINCIPAL STRESS PARAMETER $\frac{\sigma_{1,2}}{3\tau_a}$ VS. FIBER DISTANCE Z FOR DIFFERENT RECTANGULAR LOAD-SECTION PARAMETERS $C_h = \frac{M}{Vh}$

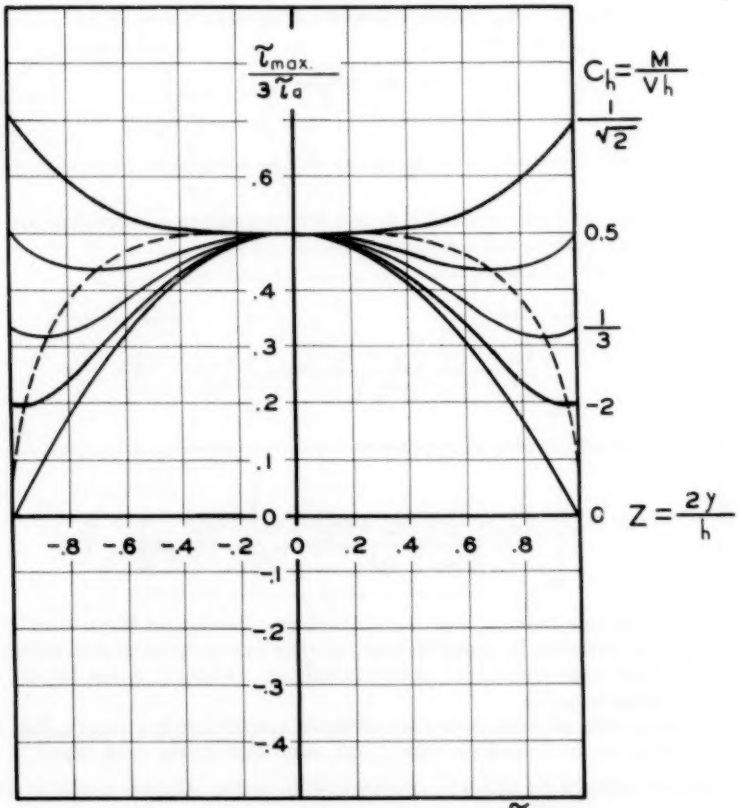

FIG. 6 MAXIMUM SHEAR PARAMETER $\frac{\tau_{max.}}{3\tau_a}$ VS. FIBER DISTANCE Z FOR DIFFERENT RECTANGULAR LOAD-SECTION PARAMETERS $C_h = \frac{M}{Vh}$

We now turn to the problem of predicting the presence of intermediate-fiber extrema along the length of a given beam. Clearly, this depends on the availability of bending moment and shearing force equations as functions of the distance along the span. Considering again the case of a rectangular section, we now examine the four simplest cases of beam loading with the object of obtaining an intermediate-extrema criterion involving both depth and span.

We first take the case of a uniformly loaded, simply supported beam, as shown in Fig. 7. The moment and shear equations are

$$M = \frac{wx}{2}\left(L - x\right) \tag{20}$$

and

$$V = w\left(\frac{L}{2} - x\right) \tag{21}$$

We now use the inequalities (16) and (17) and, after substitution and transformation, we have the desired criteria:

For principal stresses:

$$\left| \frac{(x/L)(1 - x/L)}{1 - 2x/L} \right| \leq \frac{1}{3}\frac{h}{L} \tag{22}$$

For maximum shearing stress:

$$\left| \frac{(x/L)(1 - x/L)}{1 - 2x/L} \right| \leq \frac{1}{\sqrt{2}}\frac{h}{L} \tag{23}$$

In a similar fashion, the corresponding expressions for a simple beam loaded at midspan by a single load, and for cantilevers loaded uniformly and by a single load at the free end respectively, (Figs. 8, 9 and 10) are obtained and tabulated below.

The graphical representation of these conditions is given in Fig. 11. It provides for the four elementary, but most commonly used cases, a check whether a given section at a distance $\frac{x}{L}$ from the support contains intermediate stress-extrema or not. To illustrate, assume a uniformly loaded cantilever beam, the $\frac{h}{L}$ of which is 1/2. The intersection of a horizontal line in Fig. 11 through $\frac{h}{L} = 1/2$ with the appropriate line (No. 4) shows that for $\frac{x}{L}$ 0.66 the h/L lies below the critical line, which means according to the above table that no intermediate extrema exist in this part of the beam. For values of x/L between 0.66 and the end of the cantilever, however, we may expect intermediate extrema whose magnitude and location within the section must be investigated by the method outlined earlier in this paper. Of course, values of h/L of the order 1/2 are rare. In fact the designation "beam" is probably not very appropriate in such cases, or for larger h/L. In any event, Fig. 11

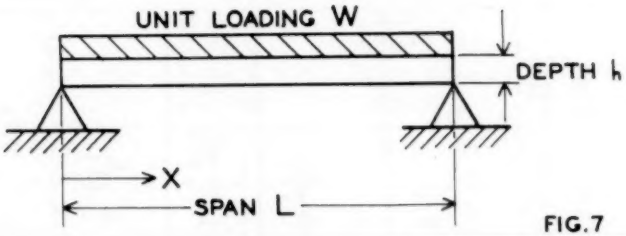

FIG. 7

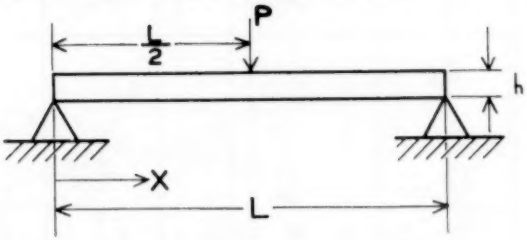

FIG. 8

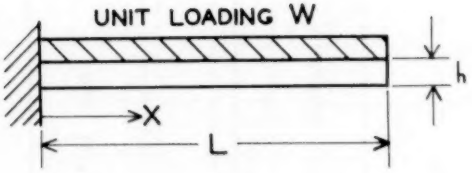

FIG. 9

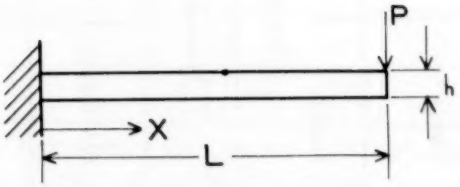

FIG. 10

CASE BEAM	LOADING	CONDITIONS FOR INTERMEDIATE EXTREMA PRINCIPAL STRESS	MAX. SHEAR STRESS
SIMPLE	UNIFORM	$\frac{x/L\,(1-x/L)}{1-2\,x/L} \leq \frac{h}{3L}$;	$\frac{x/L\,(1-x/L)}{1-2\,x/L} \leq \frac{h}{\sqrt{2}L}$
SIMPLE	SINGLE, AT MIDSPAN $0 < x < L/2$	$x/L \leq \frac{h}{3L}$;	$x/L \leq \frac{h}{\sqrt{2}L}$;
	$L/2 < x < L$	$1-x/L \leq \frac{h}{3L}$;	$1-x/L \leq \frac{h}{\sqrt{2}L}$;
CANTILEVER	SINGLE, AT END	$x/L \leq \frac{h}{3L}$;	$x/L \leq \frac{h}{\sqrt{2}L}$
CANTILEVER	UNIFORM	$1-x/L \leq \frac{2h}{3L}$;	$1-x/L \leq \frac{\sqrt{2}\,h}{L}$.

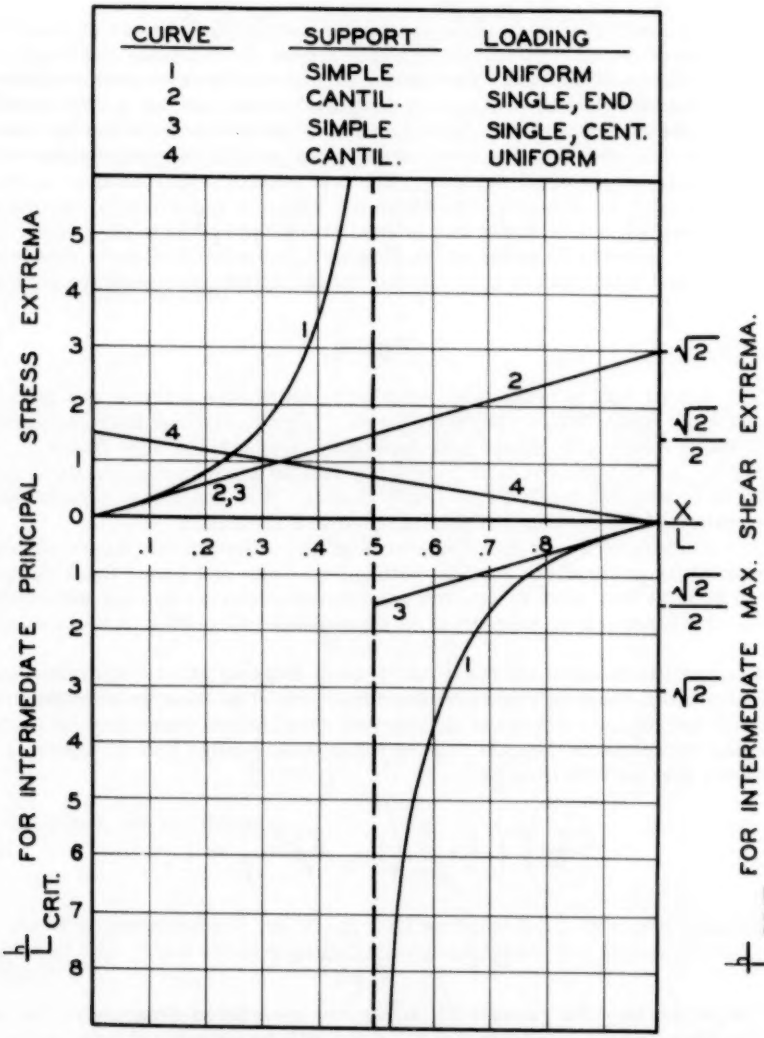

FIG. 11 RECTANGULAR BEAM CRITICAL DEPTH RATIO VS. DISTANCE RATIO.

shows also that for "normal", i.e. relatively small h/L - values, the occurrence of intermediate extrema is always confined to the vicinity of the supports where the shear is large and the bending tends to zero.

An experimental investigation was performed for the purpose of checking the agreement between observed principal stress distributions and those predicted by the above method. The testing consisted of a photoelastic examination of suitably loaded beams of rectangular cross-section, subject to different simulated end conditions. The photoelastic method was chosen because it affords an efficient means of determining principal and maximum shear stress distributions across beam sections. The test results, reported upon in detail elsewhere (Ref. 1), substantiated within the limits of experimental accuracy the distributions calculated by the method presented in this paper. In all cases, the computed locations of the principal and maximum shear stress extrema agreed within better than 3% with the experimentally obtained points.

I - Section

The second typical beam cross-section to which this analysis was applied is that of an equal-flange, straight I-beam. The calculations have been kept sufficiently general to include both American standard and wide-flange sections. As will be seen later, certain plausible assumptions had to be made in order to keep the method reasonably simple. It is hoped that experimental verification of the results can be attempted at a later date.

The notation used may be seen from Fig. 12. Although this figure obviously represents an idealized section, with corner radii and flange taper eliminated, an attempt was made to use this geometrical shape as an equivalent to the actual cross-section by suggesting an appropriate calculation of the distance h_1.

The specific criteria for the occurrence of extrema at intermediate fibers in such a beam cross-section are now developed. The basic relationships remain (7) and (8), into which the appropriate substitutions must first be made. Limiting ourselves to the case of pure transverse loading (B = 0), these equations may be rewritten as

$$\left(\frac{C}{2}\right)^2\left(\frac{k}{F'}\right)^2\left(1-\frac{2yF'}{F}\right)=1 \tag{24}$$

and

$$C\frac{k}{F'}+\frac{4F}{Cyk}=0 \tag{25}$$

The two parts of the cross-section are now considered separately: the web and the flange. New expressions for C and k will be introduced first, since they apply to the whole I - section, as against F and F' which are different for web and flange. C equals M/Vk, by definition. In order to convert this into the here more convenient form M/Vh, which again is called C_h, we multiply by h/h and have

$$C=\frac{M}{Vk}\,\frac{h}{h}=C_h\frac{h}{k} \tag{26}$$

The radius of gyration k may be obtained from the well-known formula

$$k^2 = \frac{1}{12} \frac{b_1 h_1^3 + b\left(h^3 - h_1^3\right)}{b_1 h_1 + b\left(h - h_1\right)} \tag{27}$$

As will be seen, however, it cancels out of both equations when (26) is used. The quantity F for the web is defined as Q/b_1 and is deduced from the appropriate shearing stress equation (see, for example, S. Timoshenko: "Strength of Materials," Vol. I, p. 118) to be:

$$F = \frac{b}{2\,b_1}\left(\frac{h^2}{4} - \frac{h_1^2}{4}\right) + \frac{1}{2}\left(\frac{h_1^2}{4} - y_1^2\right) \tag{28}$$

From this, for the web,

$$F' = -\,y_1 \tag{29}$$

For the flange, the assumption is now made that the shearing stress is distributed linearly in the x-direction, as shown in Fig. 13. Restricting the treatment to the vertical centerline of the beam section (y-axis), the maximum value of the shearing stress in the flange, for any fiber distance, becomes

$$\tau_2 \cong 2\,\frac{VQ}{I\,b} \tag{30}$$

From this

$$F = 2\,\frac{Q}{b} \tag{31}$$

by definition. Evaluating $Q = A\bar{y}_2$ and substituting, this gives

$$F = \frac{1}{4}\left[h^2 - (2\,y_2)^2\right] \tag{32}$$

from which, for the flange,

$$F' = -\,y_2 \tag{33}$$

These expressions for F and F', as well as those for C, are next inserted in (24) and (25). Upon simplification and introduction of the dimensionless ratios

$$R_h = \frac{h_1}{h} \tag{34}$$

$$R_b = \frac{b}{b_1} \tag{35}$$

$$R_y = \frac{2\,y}{h} \tag{36}$$

and the abbreviation

$$K = R_b - R_h^2\left(R_b - 1\right) \quad , \tag{37}$$

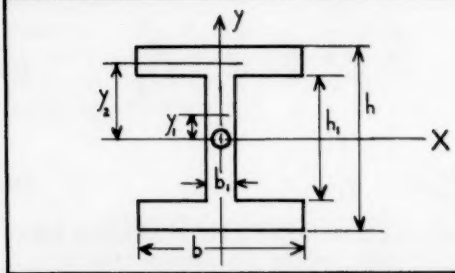

FIG. 12 NOTATION FOR IDEALIZED I-BEAM.

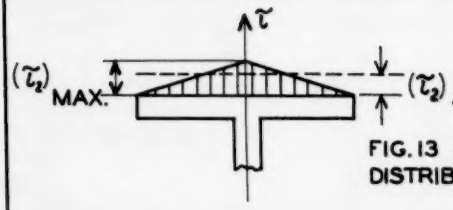

FIG. 13 ASSUMED FLANGE-WISE DISTRIBUTION OF CROSS-SHEAR.

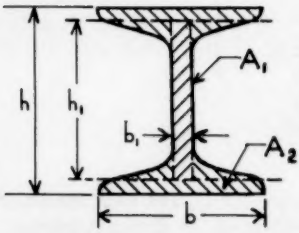

FIG. 14 ILLUSTRATING THE DETERMINATION OF AN EQUIVALENT VALUE OF h_1.

we ultimately obtain the following set of conditions:

For principal stress extrema in web

$$C_h^2 = \frac{R_y^2\left(K - R_y^2\right)}{3R_y^2 + K} \tag{38}$$

For principal stress extrema in flange

$$C_h^2 = \frac{R_y^2\left(1 - R_y^2\right)}{1 + R_y^2} \tag{39}$$

For maximum shearing stress extrema in web

$$C_h^2 = \frac{1}{2}\left(K - R_y^2\right) \tag{40}$$

For maximum shearing stress extrema in flange

$$C_h^2 = 1 - R_y^2 \tag{41}$$

While (39) and (41) could be plotted immediately since they do not depend on any web-flange ratios, an analysis of the quantity K becomes necessary when (38) and (40) are considered. By definition,

$$K = R_b - R_h^2\left(R_b - 1\right)$$

The ratios R_b and R_h as defined earlier, were first computed for a representative range of American standard and wide flange beams. While the calculation of R_b was straightforward, a special formula had to be developed for evaluating R_h. The total actual cross-sectional area A consists of the "modified" web area A_1, plus $2A_2$, where A_2 is the remaining area of one flange, as shaded in Fig. 14. Since

$$A_1 = b_1 h_1 \tag{42}$$

and

$$A_2 = b\,\frac{h - h_1}{2} \tag{43}$$

it follows, upon substitution in

$$A = A_1 + 2A_2 \tag{44}$$

that

$$h_1 = \frac{bh - A}{b - b_1} \tag{45}$$

With this,

$$R_h = \frac{h_1}{h} = \frac{bh - A}{h\left(b - b_1\right)} \tag{46}$$

The computed numerical values for the section parameters R_h and R_b show that they vary as follows:

R_h : between 0.79 and 0.93, approximately
R : between 6.60 and 15.1 for standard beams, and up to 36 for wide flange beams.

An attempt to use average numerical values of these ratios for the purpose of this investigation failed, since the deviations from the average amounted in the case of R_h to as much as 11%, and in the case of R_b to over 56%. The characteristic section parameter K is plotted in Fig. 15 against R_h for the range from 0.75 to 1.00, with R_b as a second variable within the range from 1 to 36.

Having the section parameter K available with no more effort than a few simple calculations involving handbook data and the chart of Fig. 15, the process of plotting and applying the equations (38) to (41) becomes considerably simpler. Figs. 16 and 17 represent plots of these equations for one-half of the dimensionless section depth R_y and for absolute values of C_h. This means that symmetrical and imaged curves respectively may be considered to exist in the corresponding quadrants. This formal aspect of the graphical representation needs further illustration and comment.

First, the limiting values of C_h for the web are found to be (for principal stress extrema)

$$C_h = \sqrt{\frac{K}{9}} \tag{47}$$

and (for maximum shear stress extrema)

$$C_h = \sqrt{\frac{K}{2}} \tag{48}$$

These relationships are obtained from the conditions for real roots of R_y from equations (38) and (40). Their significance is that they represent the largest values of the load-section parameter C_h which, for a given K, may result in intermediate fiber extrema. The numerical values of these limiting C_h's may, of course, also be read off the plotted curves in Figs. 16 and 17.

The practical application of these charts will now be illustrated with the aid of two examples.

Example I

Given: An American standard 6I 12 1/2 is used as a beam in such a manner that at a certain cross-section the bending moment is 2000 lb. in. and the shearing force is 500 lb.

It is desired to ascertain a) whether the largest principal stresses on the section occur at the extreme fibers, and b) whether the largest maximum shearing stress on the section occurs on the neutral axis. If not, it is desired to locate the intermediate fiber distances at which they are induced.

To solve the problem, we first extract the pertinent data for the beam section from a handbook:

$h = 6$ in.
$A = 3.61$ in^2.
$b = 3.330$ in.
$b_1 = 0.230$ in.

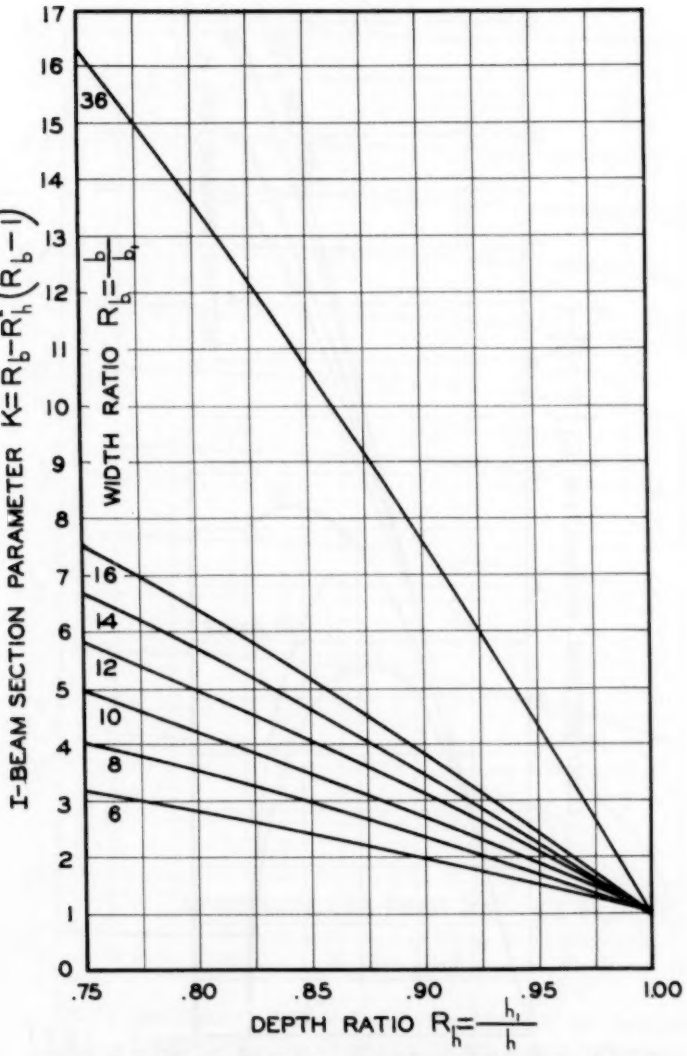

FIG. 15 I-BEAM SECTION PARAMETER VS. DEPTH RATIO FOR DIFFERENT WIDTH RATIOS. (SEE FIG. 12 FOR NOTATION)

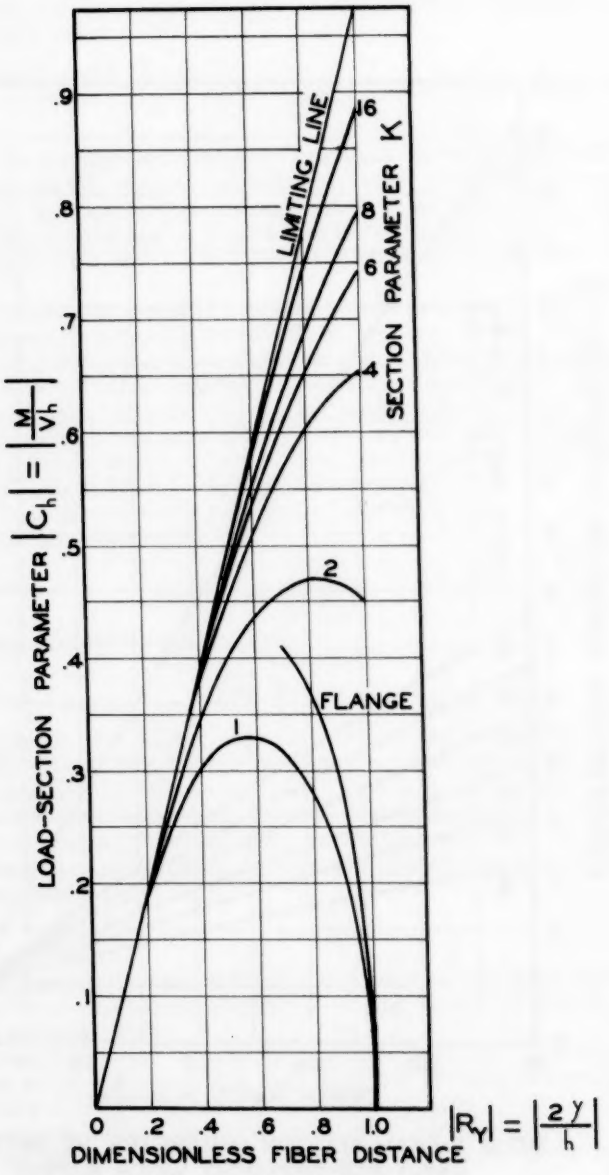

FIG. 16 PRINCIPAL STRESS EXTREMA PLOTS OF I-BEAM LOAD-SECTION PARAMETER VS. FIBER DISTANCE FOR DIFFERENT SECTION PARAMETERS $K = R_b - R_h^2(R_b - 1)$

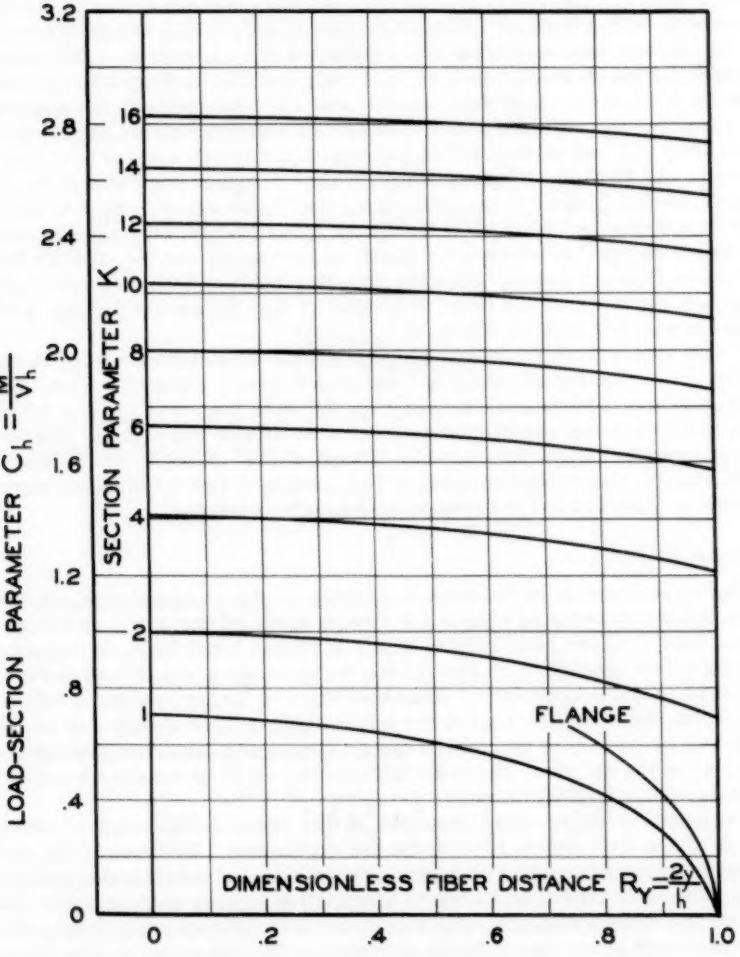

FIG.17 MAXIMUM SHEAR EXTREMA PLOTS OF I-BEAM LOAD-SECTION PARAMETER VS. FIBER DISTANCE FOR DIFFERENT SECTION PARAMETERS

$$K = R_b - R_h^2 (R_b - 1)$$

With the aid of these data we compute

$$R_b = 14.5$$
$$R_h = 0.88$$

With these ratios we enter Fig. 15 and obtain K = 4.05 by linear interpolation, checked by (37). Next, we compute the value of $C_h = 2/3$, and locate on Fig. 16 the intersection point between a horizontal line through $C_h = 2/3$ and the appropriate curve for K = 4.05. It is clear, that this intersection lies just outside the section and, therefore, no extremum at intermediate fibers exists. To illustrate an additional point, it should be emphasized that even if points of intersection had existed within the section limits, this would only have indicated an intermediate fiber extremum if the corresponding value of R_y were smaller than R_h, since beyond that value the "web-curves" cease to apply and the "flange-curves" must be used. There is, of course, no generally expressible demarkation line between the family of curves for the web and the flange curves in Figs. 16 and 17, since the boundary between them depends on R_h. However, for any specific case, this value of $R_y = R_h$ should be kept in mind when locating the critical fiber.

Continuing now with our example, we use the same values of C_h, K, and R_h in Fig. 17 for the investigation of maximum shearing stress extrema. The intersection of a horizontal through $C_h = 2/3$ with the curve for K = 4.05 obviously lies not only outside the web, but also outside the section. Moreover, the intersection of this line with the "flange-curve" gives $R_y = 0.75$ which, being smaller than 0.88 (the value of R_h), indicates that no shearing stress extremum other than at the neutral axis may be expected.

Example II

The same beam as in the first example is used for comparison. However, the moment and shearing stress are now so assumed that the parameter C_h equals 0.30, i.e., the ratio of M to V is 1.8. Since K and R_h have remained unchanged, we now get from Fig. 16 $R_y = 0.21$ on the K = 4.05 web-curve, and $R_y = 0.96$ on the flange-curve. Since the latter is larger than $R_h = 0.88$, it represents the absolute value of the dimensionless fiber distance at which there will be one set of extrema of the maximum and minimum principal stresses on the section. The fiber distance $R_y = 0.21$ gives the second set of extrema, located within the web.

The question which naturally arises at this stage is, what kind of extremum each of the so determined fiber distances represents. Mathematically speaking, an extremum may be a maximum, a minimum, or a horizontal inflection point. The well-known conditions to be satisfied in each of these three cases concern the second derivatives of the stress-fiber distance functions. While it is relatively simple to establish expressions for the purpose, they are not only somewhat cumbersome, being of the 6th degree, but also unsuitable for practical dimensionless plotting since they involve more than two parameters. It is, therefore, recommended that the inspection method be used. This method presupposes a certain amount of familiarity with the general shape of the stress distribution curves. A simple rule may be formulated thus: for relatively small values of the load-section parameter C_h, other section properties being kept constant, the principal stress curves have their significant maxima nearest the neutral axis, while for larger values of C_h these maxima gradually move out toward the flange until the abnormal condition

ceases to exist at the value of C_h which corresponds to a given parameter K (see (47) and (48), and Fig. 19). Thus, it should be possible by comparison with known distributions to label accurately each of the fiber distances obtained from the charts 16 and 17 as to what extrema they represent. (The corresponding rule for the maximum shearing stress extrema is easily deducible from Figs. 6 and 20). The actual numerical value of the largest principal stress or maximum shearing stress is then best found by substitution of the appropriate fiber distance in the dimensionless equations, obtained from (1) and (2):

For principal stress in web

$$\frac{\sigma_{1,2}}{3\tau_a} = K' \left[C_h R_y \pm \sqrt{C_h^2 R_y^2 + \left(\frac{K - R_y^2}{2}\right)^2} \right] \tag{49}$$

where

$$K' = \frac{R_b - R_h\left(R_b - 1\right)}{R_b - R_h^3\left(R_b - 1\right)} \tag{50}$$

(See Fig. 18 for plot of (50)).

For principal stress in flange

$$\frac{\sigma_{1,2}}{3\tau_a} = K' \left[C_h R_y \pm \sqrt{C_y^2 R_y^2 + \left(1 - R_y^2\right)^2} \right] \tag{51}$$

For maximum shearing stress in web

$$\frac{\tau_{max.}}{3\tau_a} = K' \sqrt{C_h^2 R_y^2 + \left(\frac{K - R_y^2}{2}\right)^2} \tag{52}$$

For maximum shearing stress in flange

$$\frac{\tau_{max.}}{3\tau_a} = K' \sqrt{C_h^2 R_y^2 + \left(1 - R_y^2\right)^2} \tag{53}$$

In Figs. 19 and 20 these equations are plotted for the beam of the example worked here, i.e. for a value of K = 4.05 and K' = 0.494, and using the corresponding limiting ranges for C as given by Figs. 16 and 17. They confirm the conclusions reached earlier concerning the location and character of the extrema, as well as the remarks regarding the overlapping of the web and flange curves. They also make it possible to obtain for any given values of C_h within the ranges considered, the ratio between principal stress or maximum shearing stress and three times the average shearing stress respectively, the latter being a convenient reference variable.

REFERENCE

A Theoretical and Experimental Analysis of Conditions Producing Principal

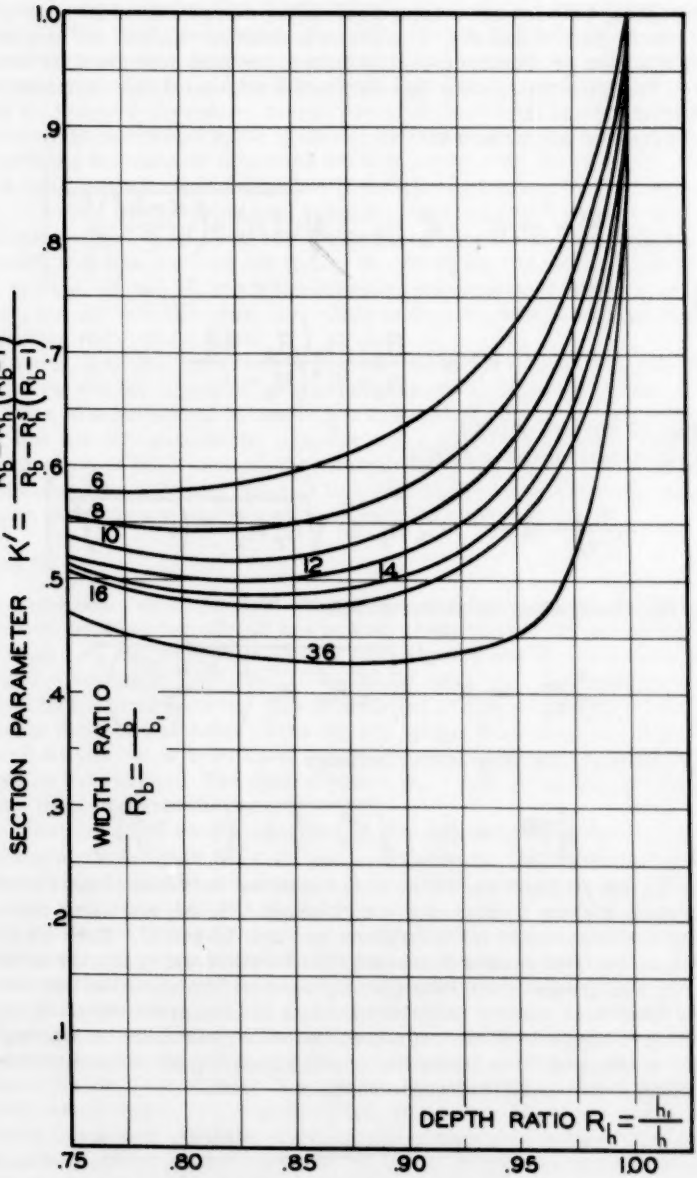

FIG. 18 I-BEAM SECTION PARAMETER K' (EQ 50) VS. DEPTH RATIO FOR DIFFERENT WIDTH RATIOS.

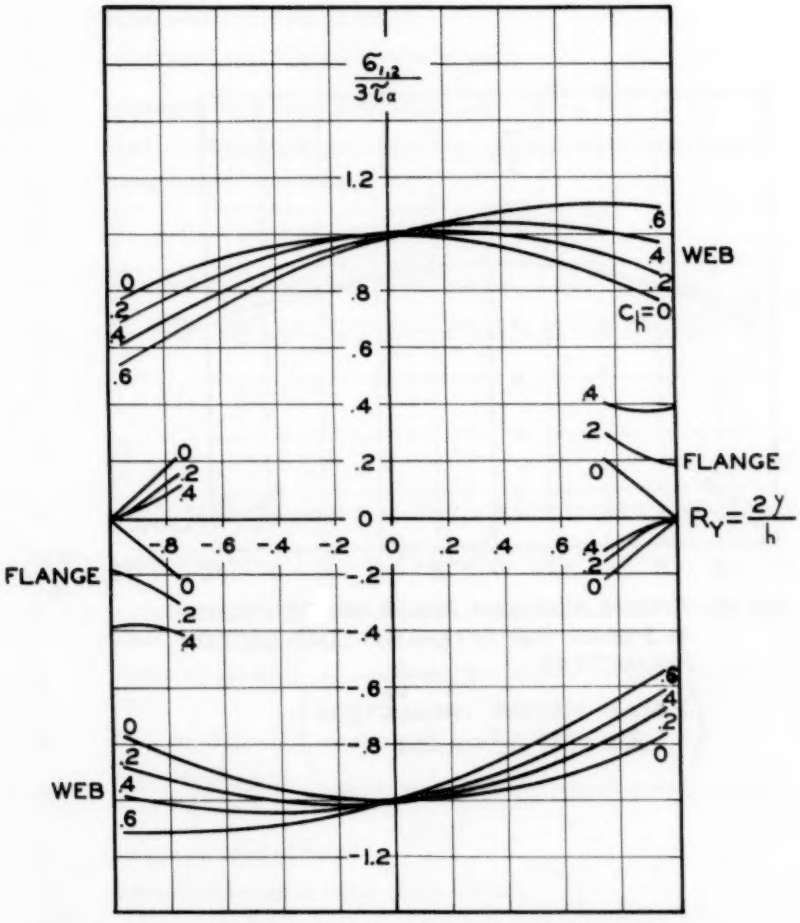

FIG.19 TYPICAL PRINCIPAL STRESS DISTRIBUTIONS IN I-BEAM FOR DIFFERENT LOAD SECTION PARAMETERS $C_h = \frac{M}{Vh}$

(I-BEAM SECTION PARAMETERS $K = 4.05$, $K' = 0.494$)

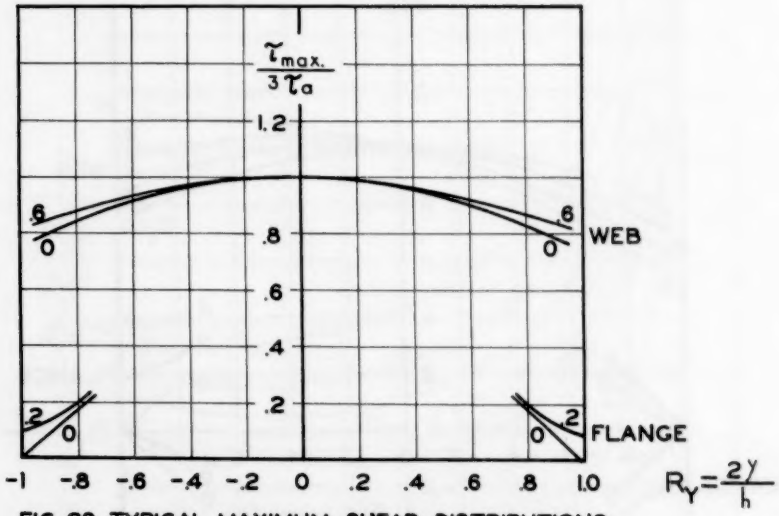

FIG. 20 TYPICAL MAXIMUM SHEAR DISTRIBUTIONS IN I-BEAM FOR DIFFERENT LOAD-SECTION PARAMETERS $C_h = \frac{M}{Vh}$

($\frac{\text{I-BEAM SECTION}}{\text{PARAMETERS}}$ $K = 4.05$, $K' = 0.494$)

and Maximum Shear Stress Extrema at Intermediate Fibers of a Transversely Loaded Beam, by Howard S. Wolko, Master of Science (in Engineering) Thesis, School of Engineering, The University of Buffalo, June 1953.

Appendix - List of Symbols

A	cross sectional area of beam
A_1	"modified" web area of I-beam ($= b_1 h_1$)
A_2	"modified area of one I-beam flange $\left[= b \frac{(h - h_1)}{2}\right]$
b	width of beam cross section at fiber y; also, width of I-beam flange
b_1	thickness of I-beam web
B	$\frac{P}{V}$ = ratio of axial load to local shearing force
C	$\frac{M}{Vk}$ = general (bending) load-section parameter
C_h	$\frac{M}{Vh}$ = modified load-section parameter, referred to depth h
F	$\frac{Q}{b}$
F'	$\frac{dF}{dy}$
h	depth of beam cross section perpendicular to neutral axis
h_1	(equivalent) height of web of I-beam $\left(= \frac{bh - A}{b - b_1}\right)$
I	moment of inertia of beam cross section with respect to neutral axis
k	radius of gyration of beam cross section with respect to neutral axis
K	$R_b - R_h^2 (R_b - 1)$ = I-beam section parameter in C_h
K'	$\frac{R_b - R_h (R_b - 1)}{R_b - R_h^3 (R_b - 1)}$ = I-beam section parameter in stress ratios
L	length or span of beam
M	bending moment in beam cross section
P	axial force on beam
Q	first moment of area beyond fiber y with respect to neutral axis
R_b	$\frac{b}{b_1}$ = I-beam flange-to-web thickness ratio
R_h	$\frac{h_1}{h}$ = I-beam web height-to-depth ratio
R_y	$\frac{2y}{h}$ = dimensionless fiber distance anywhere in I-beam section

V	shearing force in beam cross section
w	loading per unit length of beam
x	distance along length or span of beam; also direction along neutral axis
y	fiber distance from neutral axis
y_1	any fiber distance within I-beam web
y_2	any fiber distance within I-beam flange
$\bar{y}_2$	centroidal fiber distance of flange area beyond fiber y_2
Z	$\frac{2y}{h}$ = dimensionless fiber distance

δ_1, δ_2 or $\delta_{1,2}$	maximum and minimum principal stresses
δ_x	resultant normal stress in x-direction
δ'_x	$\frac{d\delta_x}{dy}$ first derivative of normal stress with respect to fiber distance
τ_a	$\frac{V}{A}$ = average shearing stress in beam cross section
τ_2	shearing stress along center line of I-beam flange
τ_{xy}	shearing stress in beam, at any fiber
τ'_{xy}	$\frac{d\tau_{xy}}{dy}$ first derivative of shearing stress with respect to fiber distance

Paper 1630 May, 1958 ST 3

Journal of the
STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

THE NEW ART OF FABRICATION ENGINEERING

Frederick H. McDonald,[1] M. ASCE
(Proc. Paper 1630)

SYNOPSIS

Fabrication Engineering is presented for a new subject in education and for the control of design in practice. It is defined as the art of determining the materials, methods and places of assembly to insure maximum use and competitive values for the products of engineering design.

In designing for structures, processes and consumer products the engineering profession and the public take it for granted that engineers are skilled in obtaining the best utility at the lowest costs. But engineers have to learn to design for other than strength, on their own—and it is easier to assume than to insure the essential choices. In our scientific age the engineer always is taught to design for safety, factually. It is unusual if he is taught to design factually for all the criteria of use—beyond strength and into costs, efficiency, service and consumer appeal.

This is the research phase of design. To be scientific it calls for the factual comparison of alternates with the integration of the best values in the materials, methods and places of assembly. Texts and publications deal with the strength of materials, as wood, metal or concrete; and with methods of fabrication, as riveted, bolted or welded connections for steel. But their costs in place, service, and the variables affecting these are seldom analyzed and less often compared in one place of reference.

Thus neither in education nor practice can the engineer generally get formulated guidance on the choices that determine the elements of fabrication. Yet these choices control the use and competitive values of all design.

Note: Discussion open until October 1, 1958. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. Paper 1630 is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 84, No. ST 3, May, 1958.

1. Cons. Engr., Charleston, S. C.

Design Status

To find what is available to insure these values in design the author queried the American Institute of Steel Construction, and the Engineering Societies Library, both in New York, as follows:

1. Is Fabrication or Fabrication Engineering treated as a special subject in engineering literature, education or practice?
2. Does the subject "Strength of Materials" treat of fabrication alternates? If so, does it deal with other relations than physical strength, such as costs, safety and service?

Or

3. Is Fabrication an unformulated subject, per se, and treated as an adjunct of structural, mechanical or other design, in education? In practice?

The Institute replied as follows:

"We know of no instance where the subject of structural steel fabrication is given special treatment in engineering literature, education or practice. Volume 1 of our Textbook on Structural Shop Drafting devotes a chapter to the fabrication of steel but this may not be as comprehensive as the discussion you apparently require."

The reply from the Library follows:

"We find no indication that fabrication is commonly treated as a special subject in engineering literature, in education, or in structural analysis. The more practical books treat types of connections, but more as separate aspects of the field than as parts of a formulated whole.

"We also find no indication that "Strength of Materials" as a subject goes beyond basic physical characteristics, and if it treats of fabrications alternates, does so selectively.

"The best we can say is . . . fabrication remains an unformulated subject. We hope this will be of some help in what sounds like a highly worthwhile project . . . for it seems that no one so far has covered the subject.

"These conclusions also are in line with the opinions of several appropriate men in ASCE headquarters."

In general, then, it seems that the scientific proofs of safety, plus scientific proofs of the best in costs and service have not been combined professionally for instruction or practice in what would amount to Fabrication Engineering. Any recognition of the importance of this will point to the need for its use to be assured by definition, formulation, publication and promotion.

Definitions

Fabrication Engineering develops proofs of the best choices in design. Its broad purpose is shown when defined as The pursuit of economies and refinements for the products of manufacture and construction. Its realism and methods are indicated by defining it as A factual comparison of the safety, cost and service values of alternate materials, methods and places of assembly for the same product, equipment or structure. The meeting of the realities of business is shown in the definition, The art of determining the

materials, methods and places of assembly to insure maximum use and competitive values for the products of engineering design.

Procedure under these definitions develops Fabrication Engineering data for a precision approach to design. As we examine the needs of all peoples for low cost products, and of business for even marginal edges in the competition of scientific marketing, we find such data so valuable that it should be made available at low costs by national collection and distribution.

Design Approaches.

Science is the key to safe performance in design, but economy is the passport to its use. Thus no matter how safe a product of design may be, if it is not within its limits of cost, it fails to be used. Yet, few engineers use a scientific approach to insure minimum costs as customarily or as skillfully as they do to minimize the hazards to safety. The proving of every assumption on stress and strain is taken for granted. But more assumptions on costs and cost relations seem to be made than are proved.

When an engineer, for instance, designs in concrete it is customary and professionally expected for him to record the formulas and calculations on his every assumption for strength. But there is no custom or professional exaction for engineers to determine and record evidence of the measure of even one alternate, say in or with steel, for the same structure. Yet this is the afterthought when firm bids exceed budgets. Here alternates are sought and usually found to lower costs. But many designs come within budgets or estimates which themselves may be excessive, because they have been assumed knowingly "on the safe side." These go into production with satisfaction—and no one knows how much less costly they might have been made to do the same or even a better job by the use of alternates that never were considered.

Cost Data and Indexes

Most engineers use some cost guides. It is customary to make preliminary estimates on the "rule of thumb" measure of the unit costs of comparable products, often designed by others at other times for other localities. Such data are the costs per square foot of office buildings, warehouses and manufacturing plants; per room of apartments, hotels, schools and hospitals; per kilowatt of installed capacity for power stations; per mile of highways; per pound or ton of gross weight for manufactured products or equipment. When such averages are current, they are useful for a "horseback" determination of financial feasibility. These generalizations, however, are hazards for conclusions on the materials and their method of assembly for a specific project.

A like hazard is in the use of historical "Cost of Building Indexes." These are composed of weighted wages and material prices for the index of a base year, which reflects the rise or fall of similarly derived indexes for any other year or period. These averages are used widely in engineering appraisals to estimate the unknown costs of previously built structures and equipment; current replacement costs; and depreciation allowances. They also are fair tools for deriving approximate current costs from older data. But it is a hazard to take them for the last word on costs.

The hazard in using averaged data for specific application is in the odds against the occurrence of the average itself among the values that have produced the average. For example, the average age of a group of a hundred people may be 40 years; but not one has to be specifically 40 years old or even

in the range from 30 to 50 if, say, the ratio of young people to oldsters is disproportionate. Such a range in the components of a cost guide could block the engineer's chances of properly shaping his design.

To illustrate, a commercial hotel with a concrete frame may have less than 10% of its area in public space and be wrong as to the cost per room or the advantage of concrete, for a resort hotel with 30% of its area in public and rentable space. Even the average cost per square foot for the commercial hotel will not be an accurate guide to the second where walls, space spans, and plumbing differ so widely in ratios to floor area. A steel or composite frame for this may be cheaper.

Short of reproduction at the same time and place, no similar product of design is likely to duplicate the average unit costs of another. "Nearly" may be a margin that can be widened on the saving side by the use of different materials or combinations, and it is this difference that can be important in the competitive market or the budget limit. Even "similar" in design may be so far from identical as to call for a more refined approach to the materials and ways of assembly. Variations can begin with the relation of productive space, weight or costs to the gross; and they range over materials and finishes, freight, labor, foundations, climate, maintenance, insurance and other factors.

The Use of Alternates

Factual findings on alternates are most valuable when used to control design. If not used and the budget is exceeded by bid prices, then there are the costs and delays of re-design. If bid prices do not exceed the budget with no study of alternates, there will be no evidence of the loss of savings that might have come from the use of alternates.

The highest values can be developed scientifically only by contrasting alternates for the current project, and not as used in other projects. This, for example, calls for determinations of concrete against steel, for a specific overpass; standard concrete against prestressed, and poured in place against precast, where it will be used; welded connections against rivets or bolts for a steel tank at its location; metal for a radio cabinet against wood or plastics, at the place of manufacture; and so on.

Places of assembly also must be compared; for the sites of structures, and the communities and even the sites of manufacture, may change costs.

Methods of assembly must be compared; for these not only vary in costs, they vary in useful life; and they vary in their capacity to develop the strengths of materials to act as a single unit. In fact, the availability or lack of a given method of assembly, such as ready-mixed concrete, or experienced riveting crews, may determine location and materials.

The Investment Factor in Design

Public works and institutional structures do not have to show an investment return, or function in competition. Yet the funds not spent for one structure by an institution leave more for another; and the less the costs of public works, the less their toll on the taxpayer. Thus these structures have their own pressures for economy. There is a personal challenge to the engineer in pride and professional repute to make his non-business designs the last word in safety, costs and service.

In commerce and industry the engineer meets an open challenge. Here price, productivity and consumer appeal are so important to profit as to exact

a factual and what might be called a "manicuring" approach to product design. Costs and consumer appeal can be ruined by generalized assumptions. Designs can be operationally effective, but profit hazards—if the engineer's assumptions have left the way open for others to use less costly alternates to make competitive equals in performance.

The Phases of Engineering

Since Fabrication Engineering is based on the need for competitive selection, it would seem fair for anyone to say that after all this is engineering itself; and the necessity for choices in design should be taken for granted. But the evidence is that too little of choice is made factually by comparative research; and too much by assumptions.

The fact that all design resolves into a physical representation of its creator's concepts and conclusions may explain why texts offer so much on the shapes and functions of materials; why publications deal so much with construction details and products; and why both have little on the design conclusions that created their pictures. This is one reason why engineers have been left to the "School of Learning by Doing," to overcome what is almost a void on all the determinations that should precede design conclusions, or the "freezing" of design.

The signs now are that the profession profitably could formulate and name the distinct phases of engineering, or the engineering approach. First would come the concept phase, to plan the organization of space and the structural controls for the accomplishment of a given purpose. This often is and properly could be called Preliminary Planning. Then would come the research phase for the devising, choice and shaping of materials, with their methods and places of assembly. This is and should be called Fabrication Engineering. Actually, whether recognized or unrecognized, it is not until these two phases are completed that sound Design Engineering can be entered for its final calculations, drawings and specifications.

Thus the use criteria of the products of design require a research and inventive operation between concept and the conclusions for design. This procedure can pay high dividends; for it is the control and the sire of the competitively poor, fair or superior design that follows.

To illustrate these phases, let us examine the separation of a grade crossing. The dimensional controls of the overpass will be determined by the identifying of traffic density, site approaches, the vertical clearance required for vehicles on the road below, the width of the lower road and its side clearances. These will set the length of the overpass, its load requirements and width, the height of its supports, and the length and grades of its approaches. These are the space and structural controls for this design; and they will provide the preliminary area, site and profile plans that can be put on paper. These preliminaries, however, are only guides to the approximate shape of the overpass and its supports; and to the approximate form and grades of its approaches.

The fabrication details of these elements will be determined by the competitive values of the materials available and their ways of assembly. This involves not only comparisons of alternates for their costs in dollars, but the computing and comparing of structural strengths and sizes for span and support variations in alternate materials. Freight charges alone may determine a major material or its shapes and methods of assembly. Soil conditions or

property costs even may dictate a change of location for the overpass and its road approaches. Then design in known materials for assembly at a stated place can proceed, with construction to follow.

Similarly in manufacture. A radio cabinet first must be conceived in terms of its space, structural and access controls for a plan of its dimensions, shape, openings and inner divisions. Then the fabrication factors must be analyzed for a choice of wood, steel, aluminum or plastics as to strength, weight, durability, appearance and costs. Design for known materials and their assembly then can proceed, to be followed by manufacture.

Thus, consciously or unconsciously, the engineer functions through the rational phases of concept, research, design and assembly. These should develop Preliminary Planning, Fabrication Engineering, and Design Engineering, to result in Construction, Processing or Manufacture. It now seems certain that the crystallization, teaching and use of each of these phases in relation to its own requirements and procedures would strengthen engineering's scientific approach to its solutions.

Science and Fabrication

Since fabrication converts nature's materials to the use of mankind, its processes always have been vital and universally in action. Today engineering determines most of these processes among productive peoples. It does seem that by now its design controls would have been formulated to meet all the criteria of use. Why do we have this omission?

Let us define fabrication as the art of choosing, shaping and joining materials in structures and products for human use. This art first emerged as a personal talent. It was transmitted by apprenticeship, to learn by doing. In time fabrication emerged as a calculable art that could be engineered by anyone informed in its principles, termed science. Instruction moved from learning by doing, to the teaching of engineering by theory.

Here constant discovery has so divided and multiplied applications that it requires a corps of academic instructors to transmit the rudiments of engineering to each student. By the pressures of volume, theory has dominated instruction. Applied or Fabrication Engineering still is left to be learned by doing—with no apprenticeships or tutors in industry for guides. A needed link still is to be forged.

Today this is a vital lack. Almost suddenly an era has come when the competition of scientifically planned mass production and its scientific marketing allow scant margin for "near misses" in the qualities and prices offered consumers. Business today has little leeway to indulge its graduate engineers with time and tolerance for adjusting theory to practice.

Yet business must cope with earnest newcomers so immersed in abstract approaches that they view design as an end in itself. Somehow these must be conformed to the reality that design is not the primary concern of commerce and industry. Here the search is for the tools of profit; and the characteristics of these tools as structures, equipment and products are important to business only as they affect productivity or salesability, against price.

Education and Practice

The arranging for engineering internes or cadetships in industry does not seem a solution. To be a gain, these still would require formalized instruction; and this still belongs with the educators. Certainly Schools of

Engineering that qualify students to shape their work to all the tests of utility can count upon turning out engineers who are superior assets in a "School of Performance" that takes safety for granted—and knows that design will fail unless it passes the cost and service tests of every facet of use.

Following are features of this subject for faculty consideration:

1. It is realistic engineering education.
2. It provides a missing link in the transition from theory to practice.
3. It is the use denominator, common to and the same for all fields of engineering.
4. It can advance science and the art of engineering.
5. It can advance professional status by making engineers more valuable.
6. It can "window dress" engineering preparation to attract more students.
7. It can make participating schools more notable.

The engineers in practice who formulate factual comparisons in costs and service for the evaluation of specific design, obtain:

1. Automatic measures of utility and price for proof of conclusive work.
2. Tangible benefits to clients and employers in money saved or better values.
3. A needed assurance of finality in economy.
4. A factual basis for decision that marks the qualified professional.

The relation of education and practice to the market place was attested long ago by Dr. Gano Dunn, late president of the J. G. White Engineering Company, in his pithy declaration to students that "Engineering is applied science under the formula, E = S + $." We would refine this by inserting CV, for Competitive Values.

Engineering Costs and Fees

Engineers will recognize that Fabrication Engineering is more than a reference to general data or pre-dated handbooks on costs. It is the measure and contrast of current facts on specific alternates for one product of design. There are now no texts or literature on these needs of designers or their experience in contrasting alternates over wide ranges in different fields of application. In the absence of these and a central reservoir of "live" data, engineers have had to and must make their own way in this phase of practice.

Only in the larger cities where a range of contractors, manufacturer's representatives and material suppliers is available can pre-bid information and costs be obtained at reasonable expense by engineers who have not collected considerable current data. In its absence for a specific design, rather than hazard assumptions, it would be in the interest of employers and clients for the significant alternates to be included in the drawings and specifications on which firm bids are taken. Here factual prices will prove relative economy conclusively. These facts may be contributed reciprocally to a central reservoir of data and held for future use.

Wherever the study of alternates is undertaken, engineering costs and time will be involved; and to the extent that such studies are more inclusive and more in detail than customary, the engineer's work will be more costly. Employers should readily grasp the benefits, however, and give approval.

In private practice where customary levels of professional fees already are being obtained, such additional time must be compensated for or become

a loss. Unless this cost can be absorbed and capitalized as an explained extra service to clients, it may be quite unsatisfactory to the engineer and the client for a higher fee to be proposed for a general design service. Here it would seem more satisfactory for the engineer to describe and agree upon the range of special studies or alternate designs he would include. For these he could make an extra charge on a lump sum or time and costs basis. It already is customary to make such charges for property surveys, soil tests and material inspections in construction; and for models and tests in manufacture. When dictated in the judgment of the engineer, charges for Fabrication Engineering studies or alternate designs would be within a range that can be justified by producing the lowest project costs or better values for the same costs.

Cautions

It should be noted that Fabrication Engineering covers more than the generally interpreted use of Engineering Economics which has been taken by many to have a different focus. The author has experienced this difference. He organized the Engineering Economics Division of ASCE in 1931. It was discontinued in 1952 because of dwindling interest by Society members—undoubtedly because of their not finding their understanding of it essential to practice or rewarding in everyday use.

"Economic Value" is a term of the economists simply for use value. This has caused Engineering Economics to be taken largely as a measure of overall value, to show investment return on a total project in operation, usually when financing is involved. The terminology also has led many to assume that it should be invoked in design only when there is an obvious and demanding choice between large or external alternates; as a site for a bridge, different routes for highways, or different communities for manufacture.

Actually, the need for the study of alternates is for proof of the why of every choice—beyond any big ones and into gains for each internal element that influences strength, cost or service. The emphasis is on more than cost. For while selling price is the control against equal competition, the offering of better service, operating or styling features can gain customer preference even at higher prices. A progressively appealing example would be air-conditioned offices—plus parking space—plus a lunch counter.

Thus engineers must work beyond the terms and values of the economists. Fabrication Engineering emerges from the language of engineers and the everyday demands upon them. They will recognize that it is more than an option that can lie "doggo" until obvious alternates cry for recognition. It is a mandate to gain the unobvious.

That this need is not theory but the utmost in realism is evidenced by the recourse of many engineers to a conscious equivalent of Fabrication Engineering; and some to an organized use. For instance, we find that the General Electric Company has an internal "Value Analysis Service" which is "a company-wide activity that is cutting millions of dollars out of product costs;" and that the U. S. Navy has a "Value Engineering Branch" for improving costs and performance in all shipyard installations, plus many contracting shipyards. We are advised by J. I. Gifford, Deputy Head Engineer of the New York Naval Shipyard that "The approved savings resulting from studies by the New York Branch (alone) amounted to over $570,000 in calendar 1957."

Engineers with any such experience can advance understanding of the gains from the factual comparison of alternates if they will contribute discussions on their practice and conclusions for the record.

We would add that while professionally associated terms are desirable in engineering, the name under which anyone practices this profitable procedure is not as important as that the concept must produce not just the static economic values of the economists, but the last word in the dynamic competitive values of Fabrication Engineering.

To illustrate: anything may have value if there is no more satisfactory alternate—the foul water of a mud hole in a desert, for instance. An industrial example would be a badly needed gage that someone makes of tin at a cost of $1.00 per pound of product, with each gage good for a year's service in tool making. The cost of $1.00 a pound per year may be a good static or economic value, when there is no competition.

But let the originator or someone else make this gage from chrome steel at a cost of even $5.00 per pound of a product that is good for 50 years of service—and the cost drops to 10¢ a pound per year, plus more precision, reliability and permanence. Here the tin gage has no dynamic or competitive position against the better and cheaper product of Fabrication Engineering.

Goals

We can rest our case on the fact that the need of all peoples, and not just Americans, is for plenty. No one can buy as a consumer beyond his earnings as a producer. Hence a more contented and peaceful world means the meeting of the cry of its worker-consumers to be able to acquire more and more satisfactions at less and less costs.

As agriculture, industry and commerce move toward the essential world goal of turning out more and more at less and less costs, they increasingly must require proof that each of their tools of production and service is the ultimate for low costs of operation. If we add that engineers more and more will offer such proofs as customarily as they now record stresses and the sections of materials chosen to meet them for strength, then we will approach the era of the most attainable plenty for everyone. Today and then, the engineers whose designs result in less and less costs will be the people to whom the world will turn to shape more possessions and conveniences.

CONCLUSIONS

The provision of safe strength and the highest service values at the least cost is essential to the survival of structures and products in their competitive markets; to the safety of financial investment; and to the attainment of world plenty. But attaining the last word in these values is more latent than achieved; for the controls that are necessary to give engineering design its highest competitive values have not been formulated professionally.

The verdict seems reasonable that the economy of the world and the satisfactions of its peoples can be improved if the education of engineers and their practice emphasize the application of such controls through Fabrication Engineering.

Such applications are and will be made by resourceful engineers. But the widespread practice of Fabrication Engineering requires 1) that a continuing reservoir of Fabrication Engineering Data be accumulated centrally; 2) that such data and its analysis be published periodically; 3) that current data be made available between publications, cheaply; 4) that the benefits from the use of Fabrication Engineering be publicized purposefully; and 5) that the

textual matter on principles, data and use be put in preparation.

It looks as if the benefits of such an undertaking would be so much in the public interest as to warrant its execution on a non-profit basis by a financially strong Foundation; by the Engineering Societies acting jointly; by a similar grouping from education; or by a combination from all of these, and business itself.

Business has a vital interest. The sale of superior products often is lost by the lack of factual comparisons for price and utility. The need is for each product to have its day in more cost and service minded courts. Here the advantages and limitations of products would be registered over wider ranges of application. Each producer would have more substantiation for advantages; and each could be guided by the research of Fabrication Engineering to offset limitations, add competitive values and further widen use.

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MOMENTS IN CONTINUOUS BEAMS ON FLEXIBLE SUPPORTS

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(Proc. Paper 1631)

SYNOPSIS

Adapting concepts previously developed by Grinter and Southwell, a procedure is presented for determining end moments in continuous beams on elastically deflecting supports. By introducing imaginary hinges at supports, a statically determinate structure is obtained. Angular discontinuities in this structure are then computed, first for separately applied unit end moments, and again for the given loading. Using this data the required moments are obtained by a converging series of successive corrections re-establishing continuity at the supports. Application is illustrated by a numerical example.

INTRODUCTION

Structures such as guyed towers and certain bridge floor systems provide examples of beams continuous over flexible supports assumed to deflect elastically. Except in certain cases, these support deflections limit the use of the ordinary moment distribution procedure, and the calculation of bending moments becomes a time consuming process, regardless of the approach used.

The purpose of this paper is to describe a procedure for obtaining these bending moments by successive corrections. The procedure eliminates any necessity for expressing the moments sought in terms of reactions, rotations, or deflections. It also avoids the serious convergence problems which afflict some other successive correction techniques when applied to the analysis of this type of structure.

The method is an extension of the process of balancing angle changes

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originated by Grinter.(1) The computation arrangement is similar to the relaxation technique developed by Southwell.(2)

Definitions

The following terms are introduced for clarity in explanation:

Angle change (ϕ) is the angular discontinuity at the end cross sections of adjacent, simply supported members at a common support (Fig. 2). The angle change consists of components due to end rotation and tilt (Fig. 3) induced by transverse loads, couple loads, end moments and support deflections. With full continuity, the angle change is zero.

Rotation (θ) is the angle change that would exist if the supports were non-deflecting (Fig. 3(a)). It should be noted that the terms, angle change and rotation, are used here with different meanings than those used previously by Grinter.(1)

Tilt (τ) is the angle change due solely to support deflection (Fig. 3 (b)).

Balancing is the reduction of unbalanced angle changes by applying equal end moments of the same sign to adjacent, simply supported members at a joint (Fig. 3).

Joint is the intersection of members at a common support.

Base structure (Fig. 2) is the structure obtained by eliminating continuity at supports of a continuous beam, thereby permitting angle changes to occur.

Spring constant (k) is the force required to displace the support a unit distance normal to the beam axis.

Sign Convention

The following sign convention is selected as being the most suitable:

Positive end moments cause compression in the upper fibers of the beam (Fig. 3).

A positive rotation or a positive tilt at a given joint has the same sense as that caused by positive end moments acting on the members framing into the joint. This is illustrated at the center support of the beam in Fig. 3. Usually, rotations have the same sign at either end of a beam. Tilts have the same numerical value at either end of a beam, but signs are opposite.

Calculation of Angle Changes

Certain basic quantities must be computed before the balancing procedure can be used. These are (1) the angle changes throughout the base structure induced by unit balancing moments applied in turn at each temporarily hinged joint and (2) the angle changes in the base structure due to the given loading.

The calculation of these quantities is a straight-forward process involving end rotations and tilts due to the unit balancing moments and those due to the given loading. The methods for obtaining rotations are well established;

therefore, they are not discussed in this paper. Tilts are obtained from the support deflections, which, in turn, are found from the statically determinate reactions of the base structure.

Balancing Angle Changes

To apply the procedure to a beam such as that shown in Fig. 1, imaginary temporary hinges are placed at certain supports, breaking continuity at these points. In this way, the beam is converted to a statically determinate base structure consisting of a series of spans with unbalanced angle changes at each hinged support. Continuity is then restored by applying end moments causing angle changes with signs opposite to those of the original unbalanced angle changes.

In this balancing operation, the end moments are applied to one joint at a time. This causes angle changes at other joints as the members undergo simultaneous rotation and tilt. The angle changes due to the balancing operation are combined with those existing previously at the various joints to give residual unbalanced angle changes, thereby completing one cycle. Succeeding cycles repeat this balancing operation at all joints in the structure where unbalanced angle changes exist. Usually such joints must be balanced several times before residuals are negligible. When convergence is satisfactory, the end moments involved in each cycle are totaled to give the final end moments.

Example

Operational details of the procedure are best described with a numerical example. The beam of Fig. 5, adapted from a preliminary study of a guyed tubular mast, is used for this purpose.

Fig. 4 shows the calculation of the angle changes resulting from unit end moments applied separately at A, B, C, and D. Shears, reactions, and deflections are successively computed. From these, the tilt in any given span follows by dividing the algebraic sum of the deflections by the span length. The algebraic sum of the tilts and end rotations at each joint is the angle change. It is worthwhile noting that, due to tilt, the end moments at a given joint cause angle changes at the two adjacent joints on either side. It is also important to note that a partial check is provided by Maxwell's law, which requires that the angle changes symmetrically located with respect to the underlined values in Fig. 4 must be equal. In this case, 10 of the 14 angle changes are equal.

To obtain numbers which are convenient to handle and to minimize writing, the values in each row are all multiplied by the common factors[3] recorded in the extreme right hand column of the tabulation. This feature is also used in the tabulations of Figs. 5 and 7.

The line sketches of the deflected structure in Fig. 4 are primarily for use in visually establishing signs, and are believed preferable to the formulation of rules for this purpose.

Calculation of end moments is carried out in Fig. 5. Simple beam angle changes are computed on the assumption of hinges at A, B, C, and D. The end moment at E affects the angle change at D by causing increments of rotation and tilt at D.

The angle changes to be balanced are recorded on line 12. The largest of these, at B, is roughly balanced on line 13. Values on this line are obtained by ratio from those on line 17 of Fig. 4. Residuals are recorded on line 14. The largest residual, at C, is balanced on line 15, using the data from line 24 of Fig. 4.

These operations are repeated to convergence in subsequent cycles. End moment increments are totaled on line 31.

It is entirely futile to apply moments of the magnitude required to exactly balance residual angle changes, because succeeding cycles will create new unbalanced angle changes. Instead, moments should be expressed as one or two digit numbers, of such magnitude as to slightly over-balance or under-balance the residual angle changes. Errors in signs can be minimized by establishing signs as an operation separate from the recording of angle changes.

Fig. 6 is an arithmetical check on the balancing procedure of Fig. 5 which amounts to combining the original unbalanced angle changes with those resulting from the final end moments of Fig. 5. The angle changes due to the final end moments are calculated from the data of Fig. 4. Residual angle changes are small. Residuals which are too large can be treated as a new set of unbalanced angle changes. By balancing these angle changes, errors in the end moments previously obtained can be corrected.

Reactions and support deflections are computed in Fig. 7. The reactions consist of the original simple beam values of Fig. 5 corrected for the effect of end moments.

As a final overall check, end moments are calculated by conventional moment distribution. Fixed end moments are those due to the support deflections of Fig. 7 combined with those originating from the uniform load. The results, shown in Fig. 8, are in good agreement with those obtained previously.

CONCLUDING REMARKS

Although the tabular arrangement used is essentially that of Southwell,(2) the technique is actually a relaxation method in reverse. The starting condition is one of zero restraint, and the quantities to be liquidated are unbalanced angle changes instead of unbalanced moments and shears resulting from artificial restraints. The values obtained in each cycle are moment increments rather than increments of rotation or deflection.

The illustrative example presented is a simple case selected with the intent of minimizing the number of considerations incidental to describing the essential features of the procedure. Because of this, the capabilities of the method are not fully demonstrated. However, variations such as spans of unequal length and variable moment of inertia, though obviously requiring more work, entail no theoretical difficulties.

The procedure has been developed primarily for finding bending moments in stiff guyed towers in cases where difficulties arise in attempting to apply the common practice of designing to a predetermined shape of deflection curve. In this application the statically determinate base structure can be used for rapid preliminary sizing of tower members and guys. A single cycle balancing operation will then often give a close estimate of the

maximum moment in the tower. Moments due to eccentricity of the guys, and secondary moments created by heavy vertical loads can be approximated with accuracy acceptable for design purposes. The effect of shear deformations of trussed web systems can be included in the design. This is conveniently accomplished by substituting an equivalent solid web for the trussed web system.

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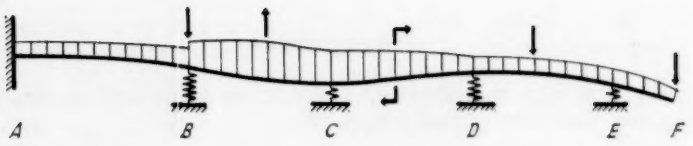


FIG. 1. CONTINUOUS BEAM ON ELASTICALLY DEFLECTING SUPPORTS

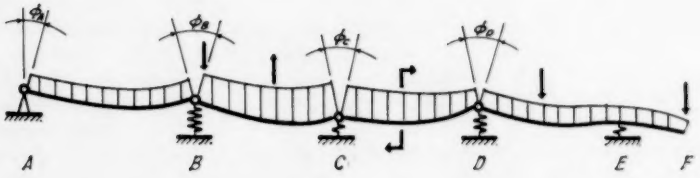


FIG. 2. BASE STRUCTURE

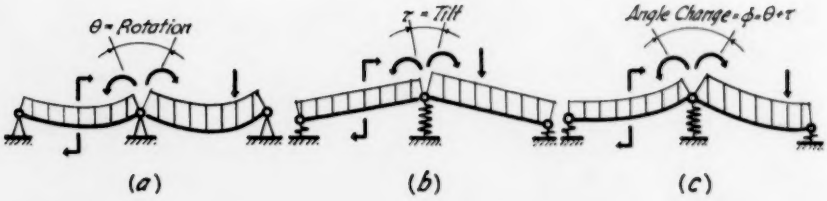
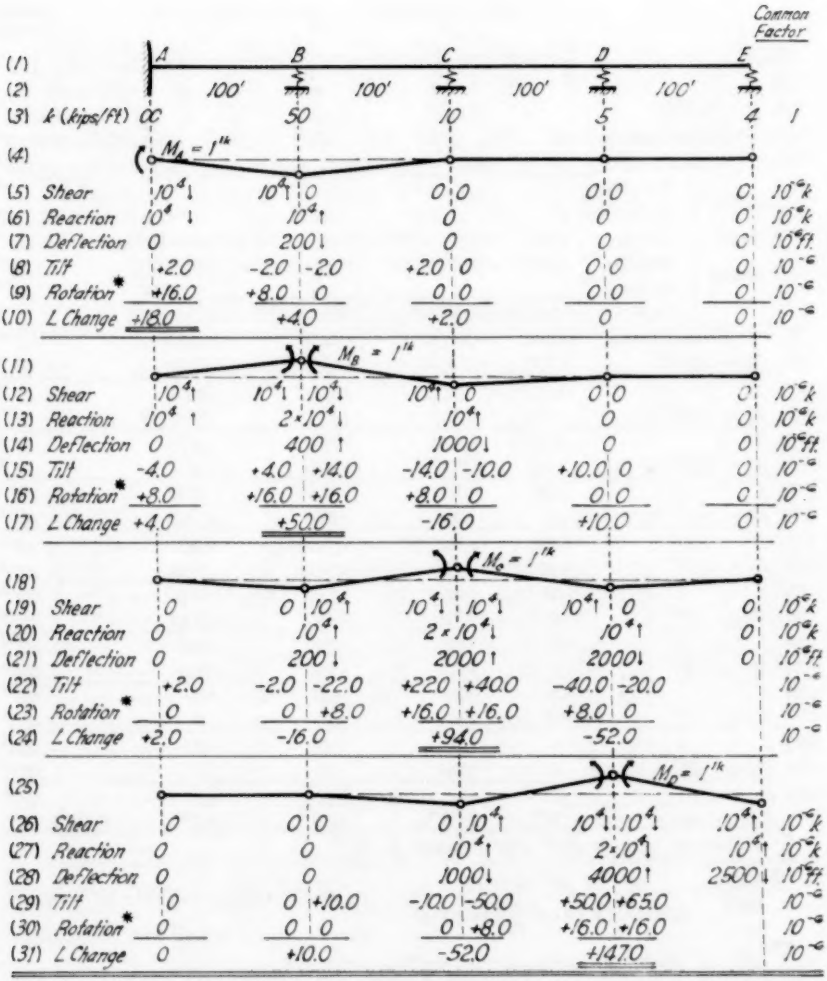


FIG. 3 RELATION BETWEEN ROTATION, TILT, AND ANGLE CHANGE

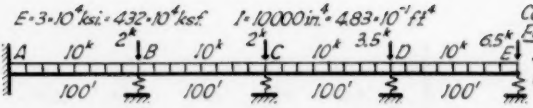


$$*\frac{ML}{3EI} = \frac{M^{tk} \times 100\ ft.}{3 \times 4.32 \times 10^4 \frac{kips}{ft^2} \times 4.83 \times 10^{-1} ft^4} = 16 \times 10^{-6} M \text{ radians}$$

$$\frac{ML}{6EI} = 8 \times 10^{-6} M \text{ radians}$$

FIG. 4. ANGLE CHANGES DUE TO UNIT END MOMENTS

$E = 3 \times 10^4$ ksi $= 432 \times 10^4$ ksf $I = 10000$ in^4 $= 4.83 \times 10^{-1}$ ft^4



										Common Factor
(1)										
(2)										
(3)	k (kips/ft)			50		10		5	4	1
(4)	Joint load reaction		0	2.0↑		2.0↑		3.5↓ 0.5↓*	0.5↑* 6.5↑	1k
(5)	Simple Beam Values	Shear	5.0↑	5.0↑	5.0↑	5.0↑	5.0↑	5.0↑ 5.0↓	5.0↓	1k
(6)		Reaction	5.0↑	12.0↑		12.0↑		13.0↑	12.0↑	1k
(7)		Deflection	0	0.24↓		1.20↓		2.60↓	3.00↓	1/ft
(8)		Tilt	+2400	-2400	+9600	-9600	+14000	-14000	+4000	10^{-6}
(9)		Rotation**	+2000	+2000	+2000	+2000	+2000	+2000	+2000	10^{-6}
(10)		Rotation***						-400		10^{-6}
(11)		Total	+4400	-400	+11600	-7600	+16000	-12000	+5600	10^{-6}
(12)		L Change	+4400	+11200		+8400		-6400		10^{-6}
(13)	$M_B = -250$ 'k		-1000	-12500		+4000		-2500		10^{-6}
(14)	Σ		+3400	-1300		+12400		-8900		10^{-6}
(15)	$M_C = -140$ 'k		-280	+2240		-13150		+7270		10^{-6}
(16)	Σ		+3120	+940		-750		-1670		10^{-6}
(17)	$M_A = -180$ 'k		-3240	-720		-360		0		10^{-6}
(18)	Σ		-120	+220		-1110		-1670		10^{-6}
(19)	$M_D = +12$ 'k		0	+120		-620		+1760		10^{-6}
(20)	Σ		-120	+340		-1730		+90		10^{-6}
(21)	$M_C = +25$ 'k		+50	-400		+2350		-1300		10^{-6}
(22)	Σ		-70	-60		+620		-1210		10^{-6}
(23)	$M_D = +7$ 'k		0	+70		-360		+1030		10^{-6}
(24)	Σ		-70	+10		+260		-180		10^{-6}
(25)	$M_C = -3$ 'k		-6	+50		-280		+160		10^{-6}
(26)	Σ		-76	+60		-20		-20		10^{-6}
(27)	$M_A = +4$ 'k		+72	+16		+8		0		10^{-6}
(28)	Σ		-4	+76		-12		-20		10^{-6}
(29)	$M_B = -2$ 'k		-8	-100		+32		-20		10^{-6}
(30)	Σ		-12	-24		+20		-40		10^{-6}
(31)	ΣM →		-176'k	-252'k		-118'k		+19'k	(-50'k)	1

* 0.5 = shear due to 50'k end moment $= \frac{50\text{'k}}{100'}$

** $\frac{WL^2}{24EI} = \frac{10^k \times 100^2 \text{ ft}^2}{(24 \times 10^4)(432 \times 10^4 \frac{\text{kips}}{\text{ft}^2})(4.83 \times 10^{-1}) \text{ ft}^4} = 2 \times 10^{-3}$ radians $= 2000 \times 10^{-6}$ radians

*** $\frac{ML}{6EI} = 8 \times 10^{-6} M = (8 \times 10^{-6})(-50'k) = -400 \times 10^{-6}$ radians

FIG. 5. CALCULATION OF END MOMENTS

(32)	Simple beam L change	+4400	+11200	+8400	-6400
(33)	$M_A = -176$ 'k	-3170	-700	-350	0
(34)	$M_B = -252$ 'k	-1000	-12580	+4020	-2520
(35)	$M_C = -118$ 'k	-240	+1890	-11090	+6140
(36)	$M_D = +19$ 'k	0	+190	-990	+2790
(37)	Σ	-10	0	-10	+10

FIG. 6. CHECK OF CALCULATED END MOMENTS

	A	B	C	D	E	Common Factor
(1) k (kips/ft)		.50	10	5	4	1
(2) Moment	-176	-252	-118	+19		1 ft-k
(3) Moment shear	0.76↓	0.76↑ 1.34↓	1.34↑ 1.37↓	1.37↑ 0.19↓	0.19↑	1 k
(4) Moment reaction	0.76↓	2.10↑	0.03↑	1.56↓	0.19↑	1 k
(5) Simple beam reaction	5.00↑	12.00↑	12.00↑	13.00↑	12.00↑	1 k
(6) Total reaction	4.24↑	14.10↑	12.03↑	11.44↑	12.19↑	1 k
(7) Total deflection	0	0.282↓	1.203↓	2.288↓	3.048↓	1 ft

FIG. 7. REACTIONS AND DEFLECTIONS

$$\frac{6EI}{L^2}\Delta = \frac{6 \cdot 432 \cdot 10^4 \cdot 483 \cdot 10^{-7}}{(1 \cdot 10^3)^2} = 1254\,\Delta''_k; \qquad \frac{3EI}{L^2}\Delta = 627\,\Delta''_k$$

$$\frac{WL}{12} = \frac{10 \cdot 100}{12} = 83''_k; \qquad \frac{WL}{8} = 125''_k$$

	A	B	C	D	E	
(1) Relative $\frac{I}{L}$	1.0	1.0	1.0	0.75		
(2) Distr		0.50 0.50	0.50 0.50	0.572 0.428		
(3) Δ (ft)	0.282	0.921	1.085	0.760		
(4) F_Δ ($''_k$)	-354	-354 -1156	-1156 -1360	-1360 -476	0	
(5) F_w ($''_k$)	-83	+83 -83	+83 -83	+83 -125	0	-50
(6) F ($''_k$)	-437	-271 -1239	-1073 -1443	-1277 -601	0	-50
(7) Bal	0	0 +629	+1258 +1258	+629 +25	+50	0
(8) Σ	-437	-271 -610	+185 -185	-648 -576	+50	-50
(9)	+220	+440 +441	+220 +350	+701 +523	0	
(10)	0	0 -143	-285 -285	-143 0	0	
(11)	+35	+71 +72	+36 +41	+82 +61	0	
(12)	0	0 -19	-38 -39	-20 0	0	
(13)	+5	+10 +9	+4 +6	+11 +9	0	
(14)	0	0 -3	-5 -5	-3 0	0	
(15)	+1	+2 +1	0 0	+2 +1	0	
(16) Σ	-175	+251 -251	+117 -117	-18 +18	+50	-50
(17)	-175	-251	-117	+18	-50	
(18) Moments from Fig. 5	-176	-252	-118	+19	-50	

FIG. 8. CHECK SOLUTION BY DISTRIBUTING FIXED END MOMENTS

Paper 1632 May, 1958 ST 3

Journal of the
STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

DESIGN FEATURES OF LOWER DECK OF GEORGE WASHINGTON BRIDGE[a]

Irvine P. Gould,[1] M. ASCE
(Proc. Paper 1632)

SYNOPSIS

The George Washington Bridge was completed and opened to traffic on October 31, 1931. The original design for this 3,500 foot suspension bridge included provisions for a second or lower deck to be constructed when traffic conditions warranted such a modification. While the original concept contemplated four tracks of rapid transit on this lower deck, the present design will incorporate six lanes of vehicular traffic with provisions for converting the two center lanes to two rapid transit tracks if that should become necessary.

This paper discusses the design features incorporated in the present structure and the modifications that will be necessary not only to the bridge, but to the New York and New Jersey approaches.

The culmination of more than four years of study and the realization of the ultimate concept of its design as visualized nearly thirty years ago was reached on July 11, 1957, when the Commissioners of The Port of New York Authority authorized sweeping improvements to the George Washington Bridge expected to cost $182,000,000, exclusive of possible Federal Aid on the New York approaches. (Fig. 1)

Scope of Project

Besides the addition of the six-lane second or lower deck to the bridge, the New York approaches (Fig. 2) will be improved to provide a twelve-lane

Note: Discussion open until October 1, 1958. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. Paper 1632 is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 84, No. ST 3, May, 1958.

a. Presented at the annual meeting of the American Society of Civil Engineers, New York, N. Y., October, 1957.

1. Engr. of Design, The Port of New York Authority, New York, N. Y.

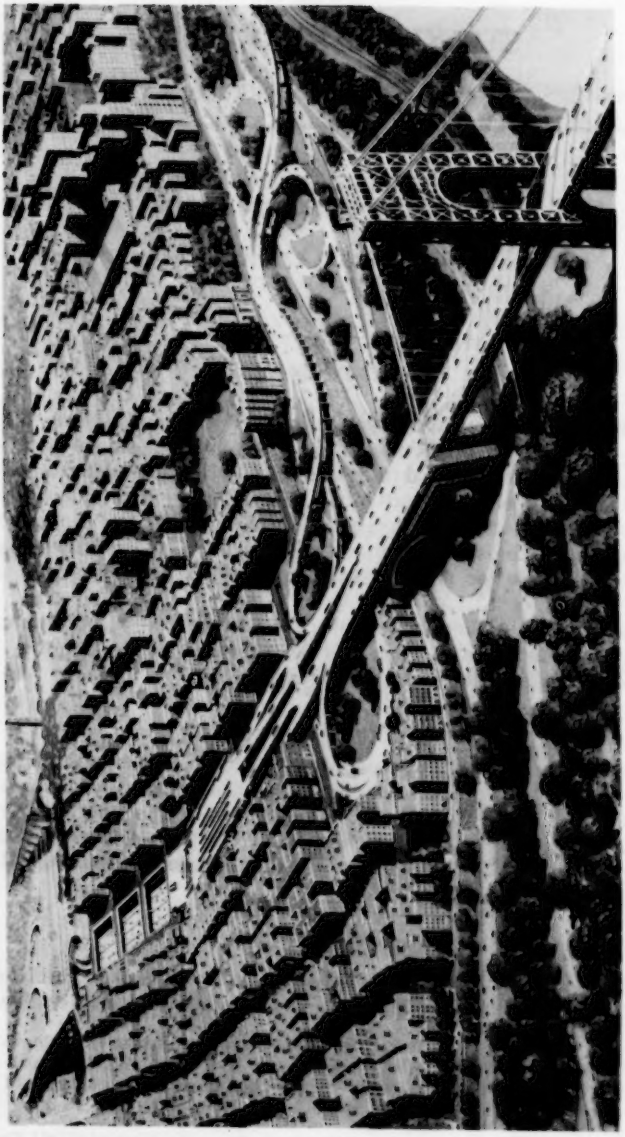


FIG. 1 - ARTIST'S RENDERING OF PROPOSED NEW YORK APPROACHES.

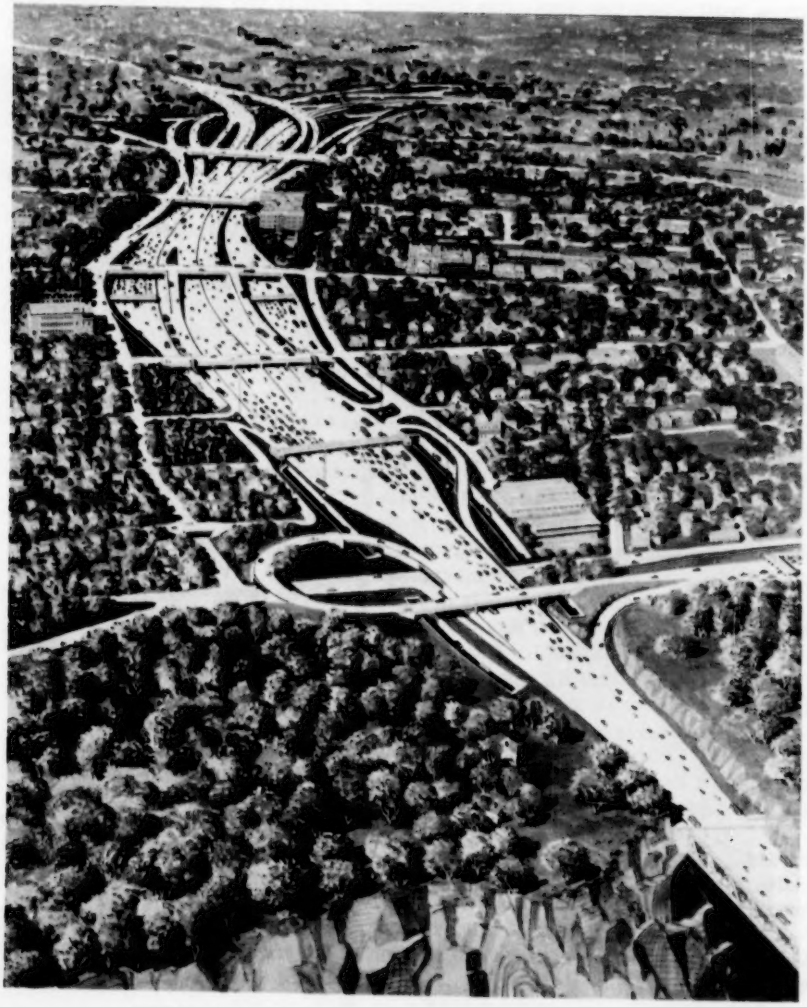


FIG. 2 - ARTIST'S RENDERING OF PROPOSED NEW JERSEY APPROACHES.

expressway which will cross upper Manhattan between West 178th Street and West 179th Street.

Astride the below street level expressway and just east of the bridge plaza will be built a new two-story bus passenger facility and an elevated bus turnaround. This will enable interstate commuter buses to drive off the bridge's upper deck directly into the passenger depot to unload patrons.

Finally, in New York, substantial improvements will be made in the approach connections to Riverside Drive and the Henry Hudson Parkway more than 115 feet below the bridge ramps.

In New Jersey, the approach improvements (Fig. 3) will include two new depressed toll plazas, two tunnels to carry vehicles on and off the lower deck, connections to the new Bergen Expressway, and the major arteries, New Jersey Route 4 and U.S. Route 46. The Port Authority will also build a new administration building and will finance and construct, and convey to the Borough of Fort Lee, after recovery of debt service charges, a 600 car municipal parking lot.

Initial Design Concept

The George Washington Bridge (Fig. 4) was opened to traffic on October 31, 1931. With a main span of 3,500 feet, it was then the longest suspension bridge in the world, but was soon to be surpassed by the 4,200 foot mainspan of the Golden Gate Bridge in San Francisco.

The bridging of the Hudson River at New York was one of those civic and engineering undertakings which for generations had attracted the ambition, talents, and efforts of many engineers, but its actual development took some fifty years to consummate. Early designs centered on various types of suspension bridges, but a 2850 foot arch and a 2100 foot cantilever received serious consideration. In its final conception, the selection of the specific suspension system and its proportions were governed by maximum economy consistent with the required degree of rigidity and aesthetic considerations.

The original design of the late 1920's (Fig. 5) contemplated seven lanes of vehicular traffic on the upper deck, but by narrowing the side lanes and rearranging the center dividing curbs, an eighth lane could be obtained. Beneath the upper deck and connected to it by rigid frames was to be erected at a later date a lower deck designed to carry four tracks of heavy rapid transit trains; or as recognized even thirty years ago by the designers, "utilized for additional vehicular lanes in case that should ever become necessary and desirable."

This decision was reached after calculations, supplemented by observations on a model, proved that the omission of the trusses and lower deck in the initial stage could still secure an ample degree of rigidity for the structure.

Upon its completion in 1931, the Bridge and approaches cost $54,860,000, including the cost of engineering, administration, real estate and financial expenses. It was estimated, at that time, that a bridge designed and built with only one deck for six lanes of traffic and two sidewalks and without provisions for a second deck, would have cost approximately $40,000,000. Of the $15,000,000 difference approximately 95% was expended for provisions for the increase in capacity of the bridge proper and only 5% for provisions in the bridge approaches. In retrospect, this additional outlay was more than well spent.

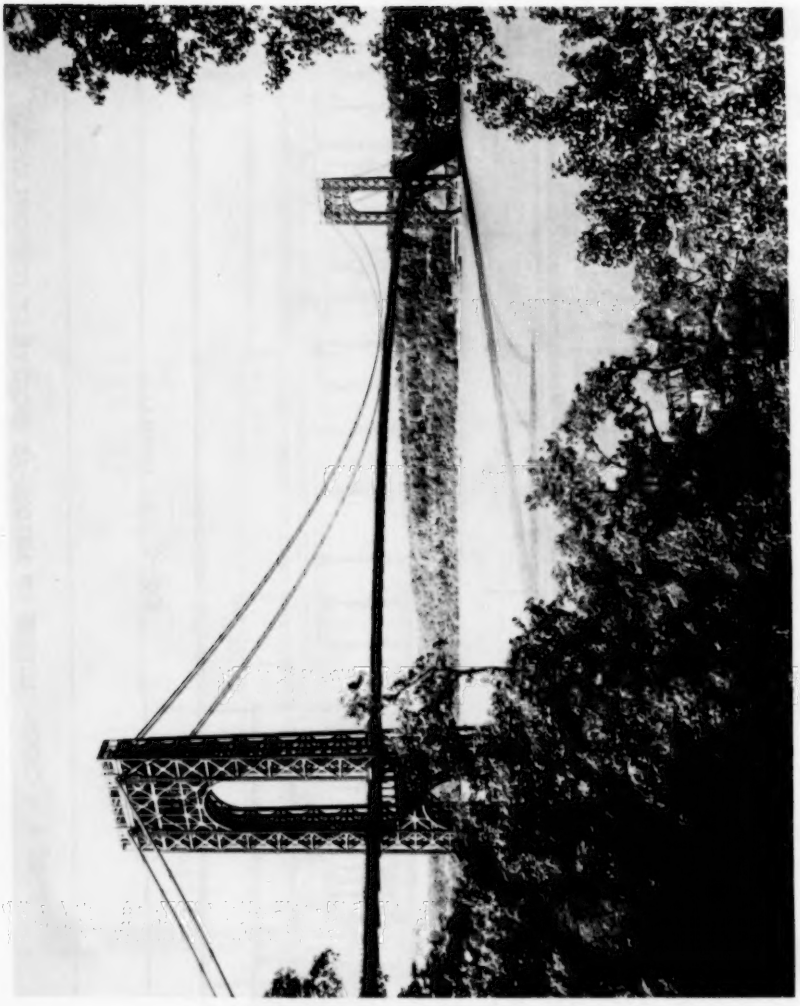


FIG. 3 - GENERAL VIEW OF GEORGE WASHINGTON BRIDGE.

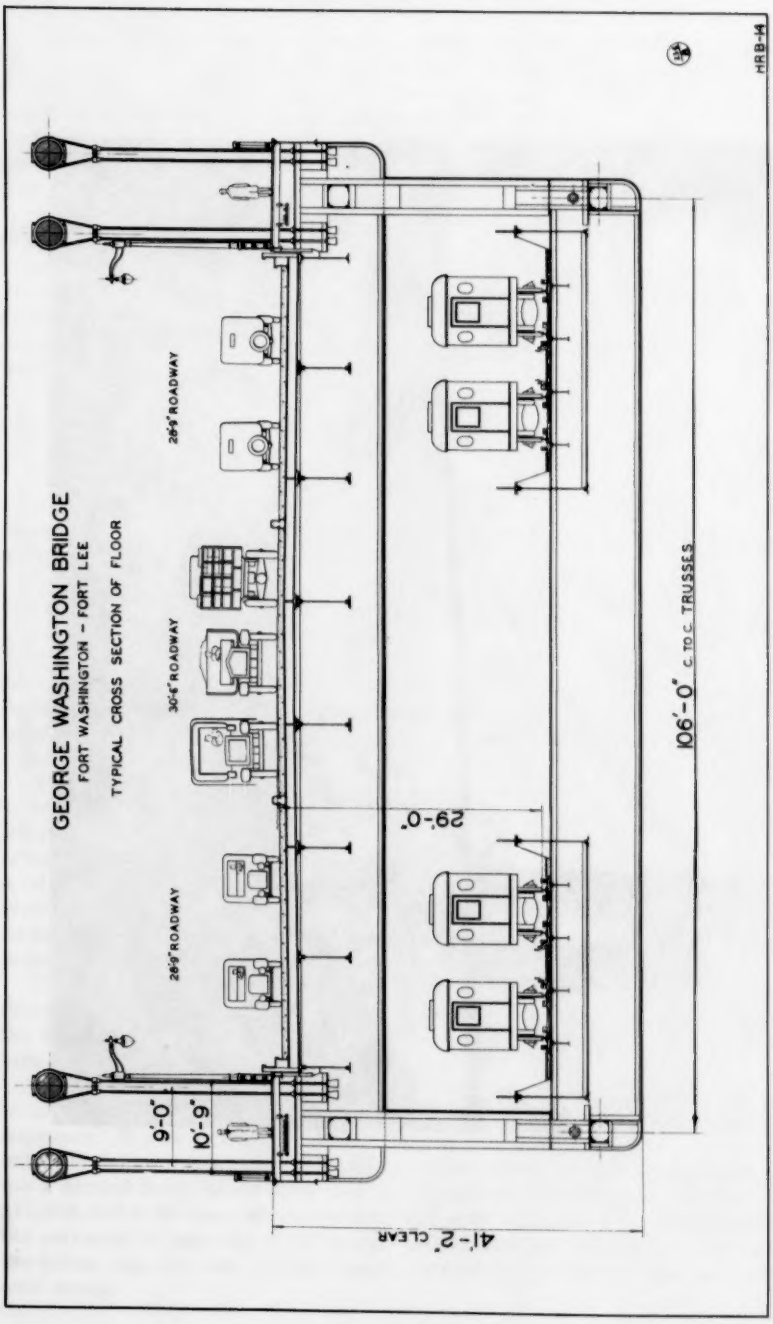


FIG. 4 - CROSS-SECTION OF BRIDGE AS ORIGINALLY CONTEMPLATED.

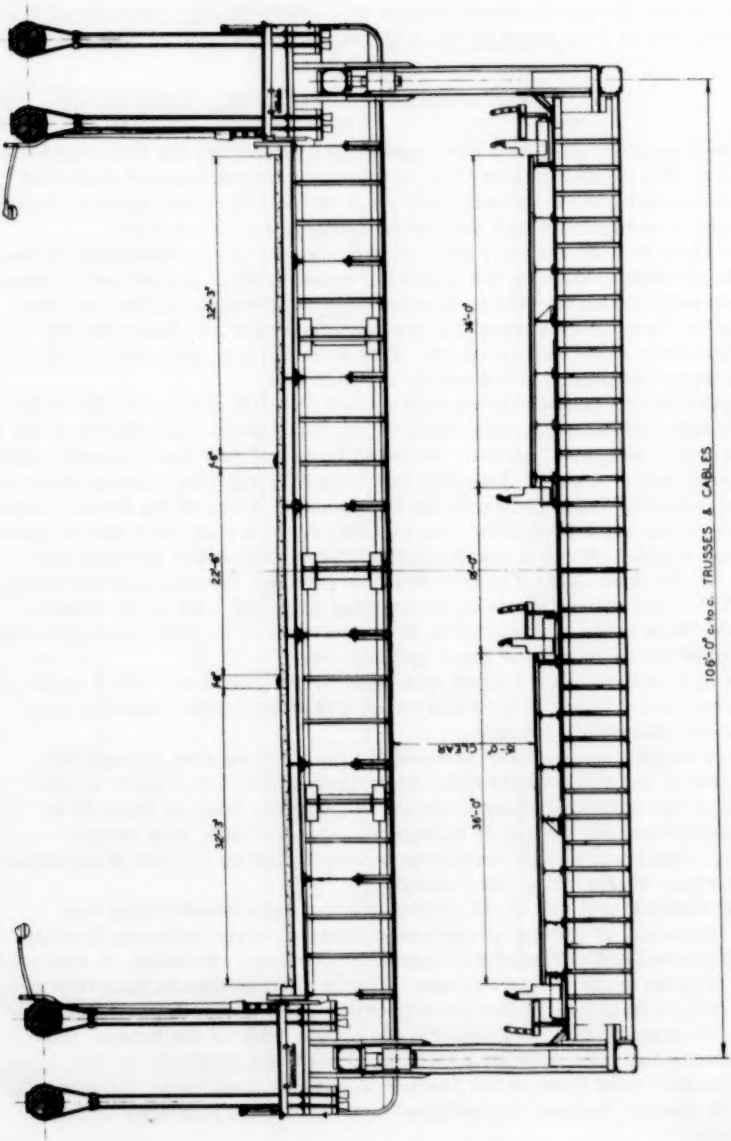


FIG. 5 - CROSS-SECTION OF BRIDGE FOR SIX LANES OF VEHICULAR TRAFFIC ON LOWER DECK.

Today this major arterial facility carries nearly 36,000,000 vehicles and is about to undergo what may be its final alteration, almost as conceived some thirty years ago. The modifications necessary to provide additional capacity for the bridge structure proper will follow closely the original plan; the approaches on both sides of the river will undergo radical changes.

Design Features of Lower Deck

As noted earlier, the lower deck was initially designed for four rapid transit tracks. The present design (Fig. 6) provides for six lanes of vehicular traffic with provisions to permit conversion of the two center lanes to double track rapid transit use should such be necessary at a future time.

In order to provide for the future construction of this second deck, it was necessary to design initially the stiffening trusses, floor system and bracing in considerable detail as well as to determine the intensity of the live load.

Based on these calculations, the assumed dead plus live loads for the completed structure was 47,000 #/lin. ft. This included approximately 3,100 #/lin. ft. for a future wearing surface on the upper deck.

The present design anticipates a total dead plus live load of 45,500 #/lin. ft. when the lower deck carries six lanes of vehicular traffic, and 46,300 #/lin. ft. when the lower deck carries four vehicular lanes and two rapid transit tracks.

In recent years the Port Authority has been investigating various types of skin coat installations for resurfacing the concrete decks of its bridge facilities. One of the most promising found to date is a one-half inch surfacing of silica sand asphalt, which if placed on the upper deck, would add only 500 #/lin. ft. to the dead load. Thus, it would be possible to resurface the upper deck and still not exceed the total design dead plus live load of the bridge.

In order to achieve this objective, it was necessary to effect economies in the weight of the floor and the floor system.

The 4-1/4 inch steel grid filled with light weight concrete, with 2 inches of asphaltic concrete surfacing rather than a less expensive concrete floor provides one source of economy.

Further weight savings were achieved in the floor system through the introduction of an intermediate floor beam at the lower panel point of the diagonals of the stiffening truss reducing the stringer spacing from 60 to 30 feet and permitting the use of stringers of considerably less weight.

Finally, additional weight reductions were effected by the use of aluminum for the service walk grating and railings.

It is of interest that one of the studies for the floor construction was directed to the use of precast prestressed concrete slabs to be set directly on the floor steel and surfaced with asphaltic concrete. However, it was not possible to achieve the weight desired. If such a design can be properly detailed to assure firm bearing on the supporting steel and with provisions for adequate tie-down, it may well become the bridge floor of the future. The Port Authority has used precast prestressed slabs successfully on two bridges for the Third Tube of the Lincoln Tunnel. In each case, however, there were specific factors that weighed heavily to the advantages of this construction.

The six-lane lower deck floor system has been detailed so as to require a minimum of modification for conversion of the two center lanes to two rapid transit tracks (Fig. 7). This can be done at a later date, if necessary, by

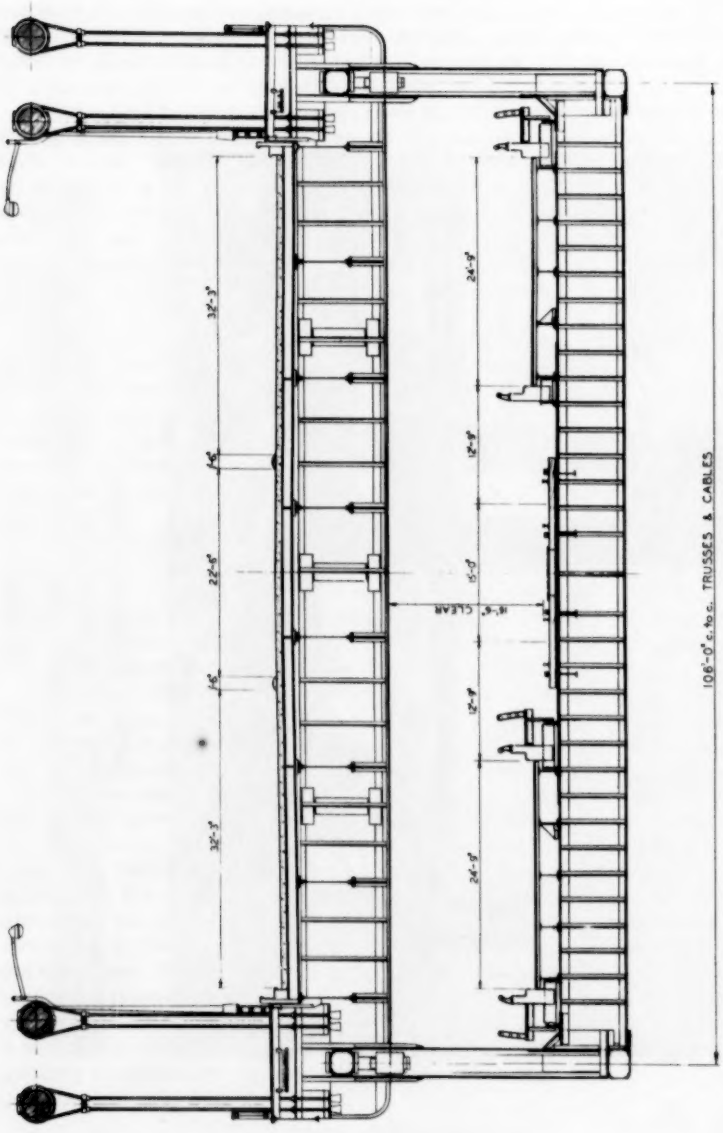
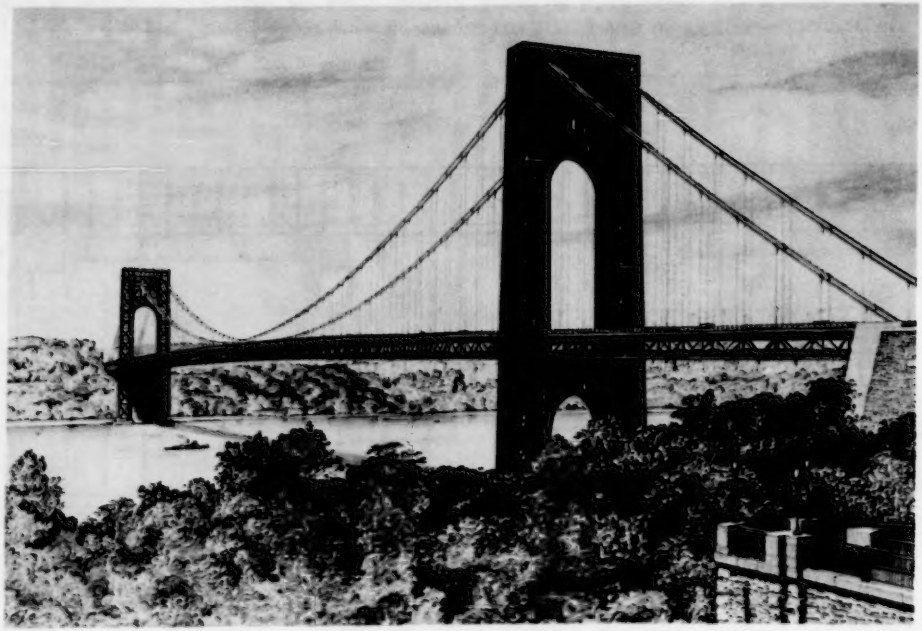


FIG. 6 - CROSS-SECTION OF BRIDGE FOR FOUR LANES OF VEHICULAR TRAFFIC AND TWO RAPID TRANSIT TRACKS ON LOWER DECK.



GEORGE WASHINGTON BRIDGE

SKETCH SHOWS THE SIX-LANE LOWER DECK AS IT WOULD APPEAR BENEATH THE PRESENT EIGHT-LANE ROADWAY OF THE GEORGE WASHINGTON BRIDGE WHICH CONNECTS WEST 178TH STREET IN MANHATTAN AND FORT LEE, NEW JERSEY.

FIG. 7

removing the floor deck in these lanes and rearranging the stringers.

In detailing the top chord of the stiffening truss, the original designers provided rivet holes, temporarily filled with bolts, for connection of the verticals to the present floor beams. In turn, gusset plates were also provided for connection of the diagonal web members. These, however, were left blank and are to be drilled in the field.

In the initial upper deck construction the wind truss consisted of the upper chords of the future stiffening trusses and diagonal bracing in the plane of these chords. It had not been planned to provide a bottom lateral system in the double deck construction, the assumption being that the wind load on the lower deck would be transmitted to the upper deck by the verticals of the stiffening truss, the verticals acting with the upper floor beams as a stiff frame. Experience has indicated the desirability of a lower lateral system in addition to the upper one. It is therefore planned to provide a lower wind truss in the plane of the bottom chord.

Since the towers were designed for the total ultimate load of the bridge, the only structural modification that may be necessary would be the framing to take the wind shear from the new lower deck wind truss. However, it will be necessary to adjust the relationship of the saddles to the center line of towers to compensate and reduce the eccentricity of load as the towers move riverward under the added load of the second deck. This will be done by jacking between the saddles and the towers to roll the towers shoreward approximately 13 inches. In their final position, the saddles will be approximately five inches shoreward of the center line of towers.

As noted previously, the stiffening trusses were designed in some detail during the original design period so that proper provisions could be made for the ultimate condition. These calculations were carefully stored and are now available for review and checking by the present designers. However, some modifications must be made for the new dead loads.

The laborious "cut and try" methods for the deflection of the cable and the stiffening truss, the long tedious equations to determine the horizontal component of the cable pull and other complicated computations necessary to design a suspension bridge and in this instance, the determination of maximum permissible distortion on the upper deck during erection of the lower deck, are ideally suited to analysis by a digital computer. Such a program is now being developed using the Bell System for programming an IBM 650 digital computer. When completed, the program will furnish data not only for the design, but for checking erection conditions. Present plans contemplate setting up the program so that it can be integrated with the erection programs developed by the major bridge companies and would be made available to them for their use. This should not only free the designers from the tedium of laborious calculations and cross checking, but should result in a considerable reduction in engineering manpower by the steel fabricator and be reflected in a savings in cost for the steelwork. It is not planned to make this program publicly available for indiscriminate use.

Criteria for Operation

Three basic criteria were set up for the operation of the bridge that govern the layout of the approaches. These were: (1) the two middle lanes of the upper deck would continue to be reversible so that a total of eight lanes of

the two level fourteen lane structure would be available for traffic in the preponderant direction; (2) approaches in New York and New Jersey would be designed so that the motorist could reach his desired route from either the upper or lower deck; and (3) the design would permit all traffic to be routed to the upper deck should this be desirable at any time, such as during periods of low traffic volume.

New Jersey Approach Modifications

Mention was made earlier of the proposed modifications to the New Jersey approach. The addition of the lower deck will require on-bound and off-bound approach roads (Fig. 8) to this level as well as additional toll booths, expanded connections to Palisade Interstate Parkway and a new administration building. The basic concept, to provide for maximum flexibility in routing vehicles to and from the bridge and the major arterial highways to either the upper or lower decks, calls for two new depressed toll plaza areas flanking the existing plaza; the plaza to the south for eastbound vehicles and the plaza to the north for westbound vehicles. These plazas will be reached by short ventilated tunnels (Fig. 9) under the present plaza near the New Jersey anchorage. As far as possible, this expanded approach will occupy undeveloped land, a part of which is now owned by the Port Authority and the New Jersey State Highway Department, with minimum acquisition of private property in the Borough of Fort Lee.

In determining the number and type of toll booths for the depressed plaza (Fig. 10), reference was made to an operations research study of Traffic Delays at Toll Booths.(1) This study, made in 1952-53, clearly demonstrated that, capacity-wise, left-hand collection booths are far superior to a combination of right and left-hand booths during peak operating conditions and under all combinations of traffic composition (percentage of passenger vehicles, buses, and trucks). A standard handling time per vehicle has been fixed at eleven seconds and, of course, varies about this mean to some extent due to human characteristics. During recent reconstruction of the toll plaza at the Lincoln Tunnel to accommodate traffic for the third tube, it had originally been planned to install a combination of 22 right and left-hand toll booths. As a result of the research effort, it was found only necessary to construct 18 left-hand booths which resulted in a substantial saving in construction costs.

This study also developed superior scheduling techniques so that the service standard of eleven seconds could be maintained with a minimum number of toll booths in operation - at an estimated operational savings for the present plaza booths of $50,000 per year.

New York Approach Modifications

The New York approaches to the lower deck (Fig. 11) will provide interchange with the Henry Hudson Parkway and Riverside Drive as well as to the local Washington Heights area by means of an extensive modification of the existing approaches. Such connections will be constructed of elevated ramps or roadways on fill. Engineering of these approaches tax the ingenuity of the designer to accomplish the construction with an absolute minimum of interference to the high density traffic traversing the present approach connections.

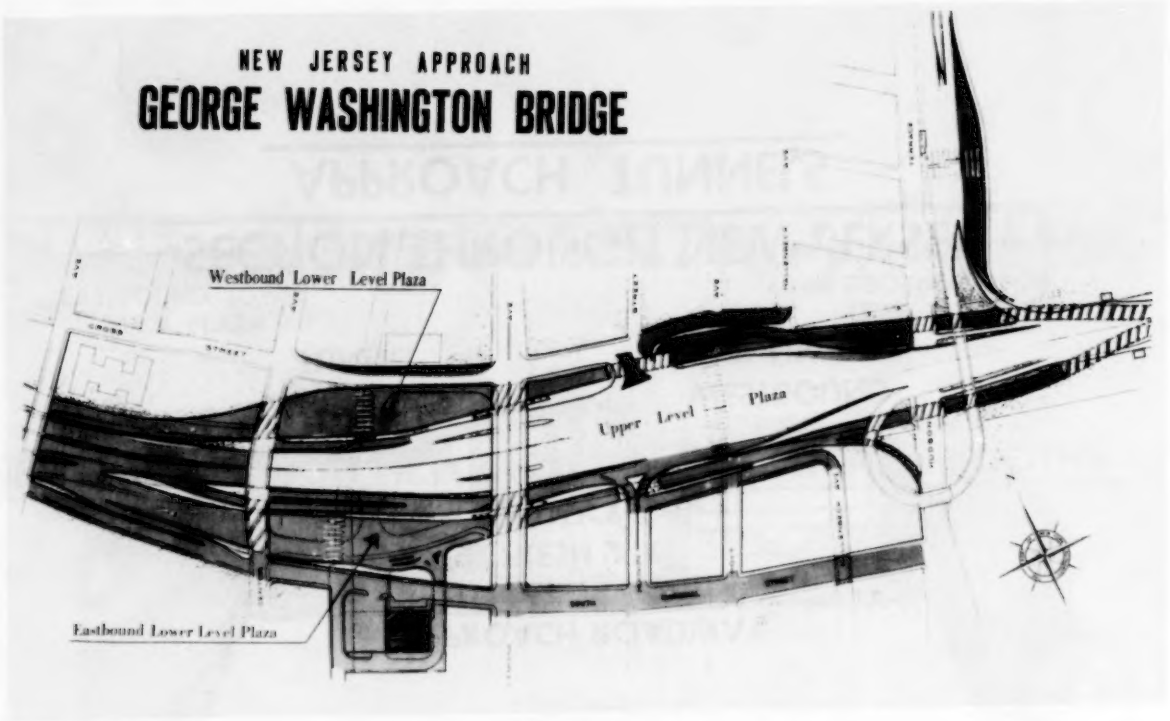
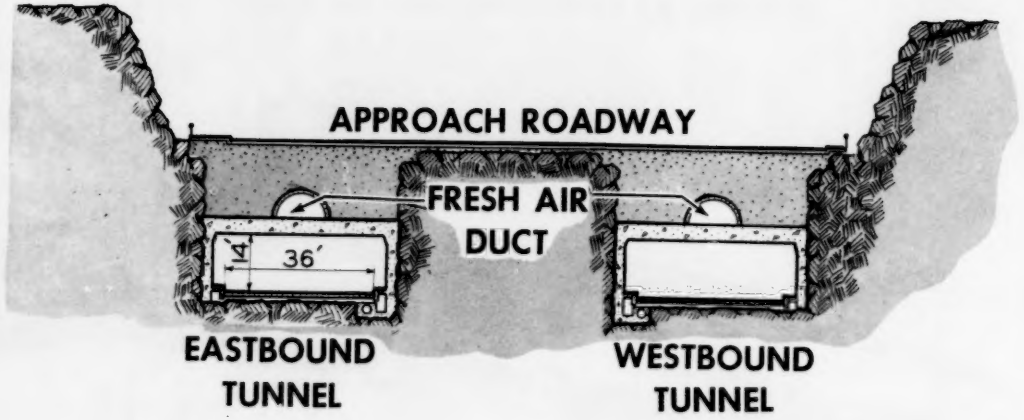
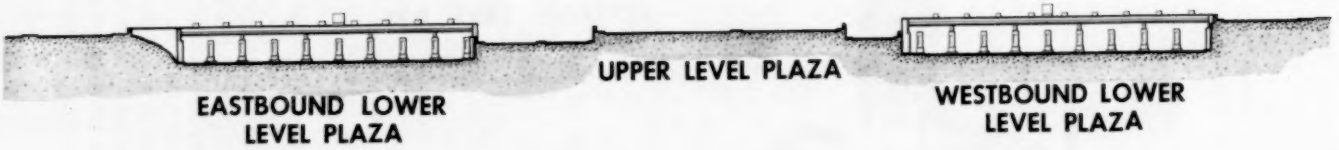


FIG. 8 - NEW JERSEY APPROACH SHOWING NEW CONSTRUCTION.



SECTION THROUGH NEW JERSEY APPROACH TUNNELS

FIG. 9



SECTION THROUGH NEW JERSEY PLAZA AT TOLL BOOTHS

FIG. 10

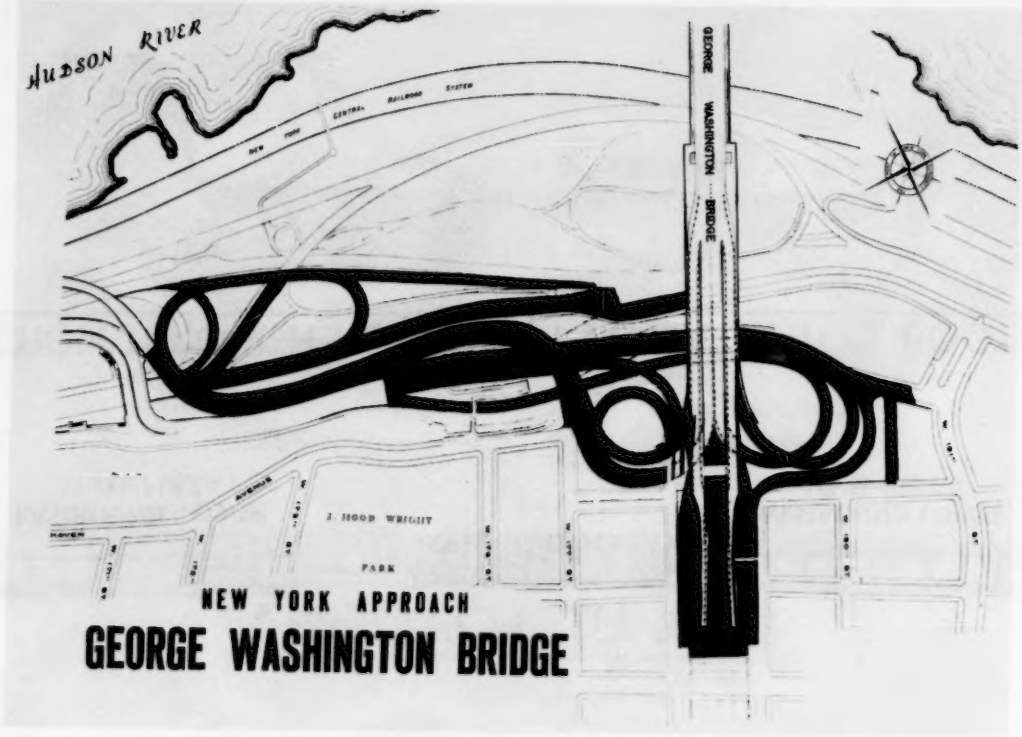


FIG. 11 - NEW YORK APPROACH SHOWING NEW CONSTRUCTION FOR CONNECTIONS TO RIVERSIDE DRIVE AND HENRY HUDSON PARKWAY.

So complex is the problem of scheduling the construction and visualizing the relationship between traffic and completed areas, work to be done and clearances in the many levels of the approaches, that it has been necessary to construct a special model for staging the construction. This model will not only supplement the scheduling of the contracts, but will give the operating staff a clearer picture of traffic flow and thus facilitate vehicle movement through the construction area.

Express traffic across Manhattan to and from the George Washington Bridge is now handled by the tunnels in West 178th and West 179th Streets (Fig. 12). These tunnels will be closed and their ventilation buildings demolished for a new twelve-lane depressed expressway between the above-mentioned streets. This expressway (Fig. 13) at its eastern end will connect with the Cross-Bronx Expressway by means of a new Harlem River Bridge just south of the present 181st Street Bridge. Direct connections would also be provided in Manhattan with Amsterdam Avenue and the Harlem River Drive.

In the design of the expressway provision will be made for development of air rights above the expressway so that at a later date apartment houses may be erected to supplant those being demolished.

Bus Facility

By agreement between the Port of New York Authority and the City of New York, the Port Authority will construct and operate a bus facility just east of the bridge plaza between Fort Washington Avenue and Broadway and above the expressway. This structure will serve the New Jersey buses that now traverse the city streets by circuitous routes to reach the local subways. Direct ramps from the upper deck to the bus level (Fig. 14) and an elevated storage and turnaround east of Broadway will now relieve the Washington Heights streets of this heavy interstate suburban bus traffic. For convenience of the bus patrons, direct underground passage will connect with the Eighth Avenue Subway.

The layout of the terminal is based on experience gained from operation of the Port Authority Bus Terminal in mid-town Manhattan. Passengers will enter the terminal either from the subway connection, Fort Washington Avenue, or Broadway and be directed by signs on the main concourse to stairs leading to the various bus islands on the bus concourse above. On the main concourse (Fig. 15) will be located the waiting room, ticket counters, and other public facilities as well as a snack bar and shops for the hurrying commuters' convenience.

Another operations research study is currently being made on the optimum design of the bus islands.

It will be based upon the operation of the suburban bus level of our mid-town terminal under peak operating conditions. This study, which will be published later this year, analyzes two converging inter-related waiting lines - one of buses and one of passengers (or trains, or planes, and so forth). Short bus islands (2 or 3 loading positions) provide the most desirable service from the standpoint of both the bus operators and the passengers, and decreases the necessity for holding or yard space. It will provide the designers with an area of study in which the relation between the platform sizes, operating roadway requirements, and hold or storage areas can be

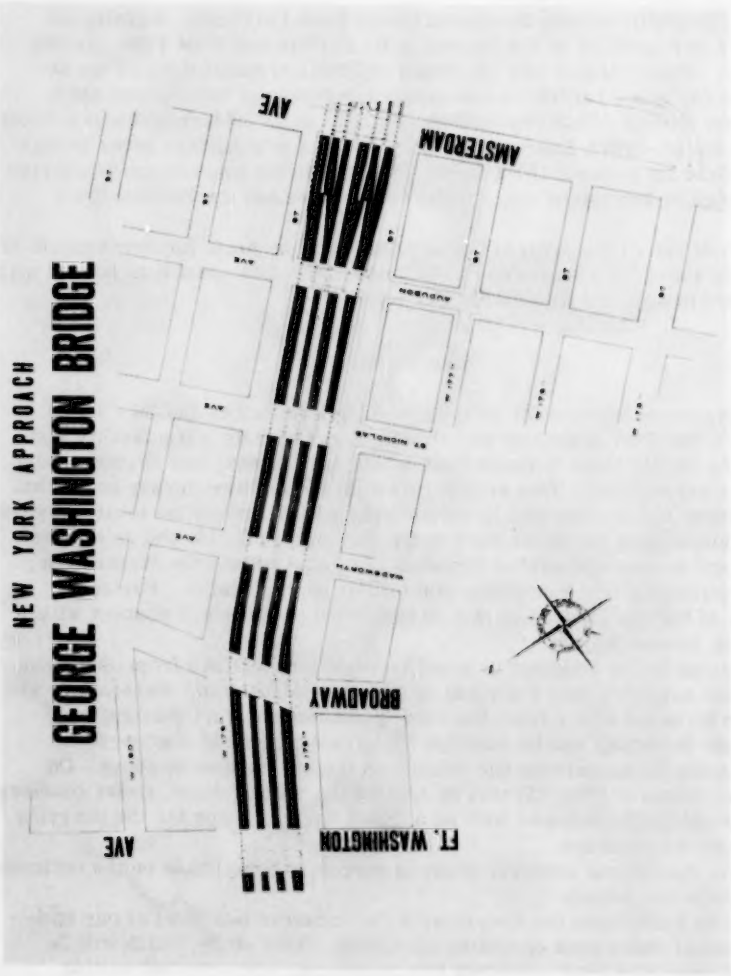


FIG. 12 - NEW YORK APPROACH SHOWING NEW CONSTRUCTION (TRANS-MANHATTAN EXPRESSWAY).

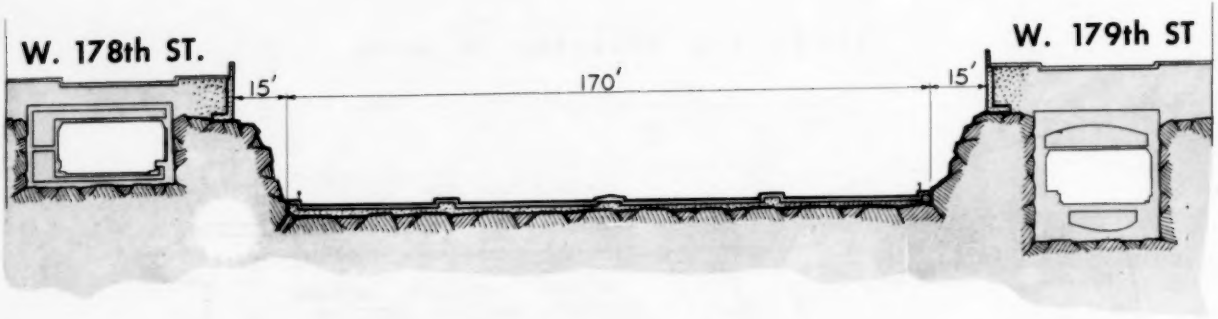
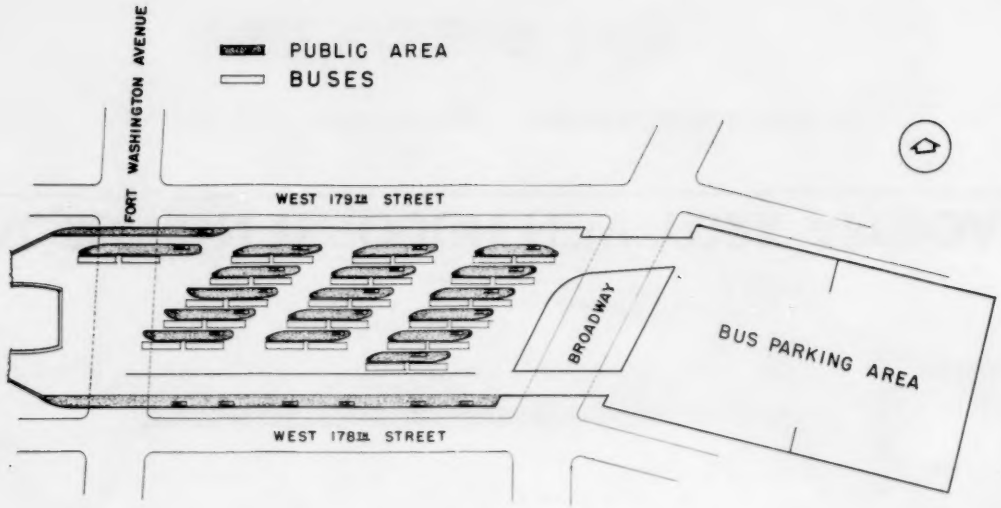


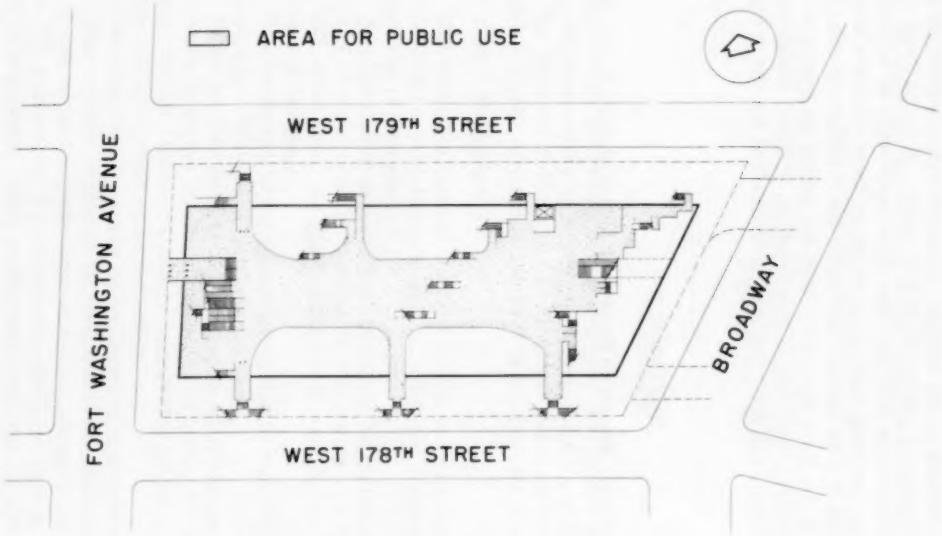
FIG. 13 - CROSS-SECTION - TRANS-MANHATTAN EXPRESSWAY.



PLAN AT SUBURBAN BUS LEVEL

GEORGE WASHINGTON BRIDGE
BUS PASSENGER FACILITY

FIG. 14



PLAN AT CONCOURSE LEVEL

GEORGE WASHINGTON BRIDGE
BUS PASSENGER FACILITY

FIG. 15

evaluated in terms of the resultant patron service. It will permit them to select an arrangement of the overall facility operating area which will be sound economically, feasible of construction, and at the same time, permit an efficient operation.

The present program, as outlined, is estimated to cost $182,000,000 or three times the cost of the original bridge when it was opened in 1931. If it had not been for the foresight of the earlier engineers and planners, in all likelihood, it would have been impossible or too costly. Some of those engineers who worked on the design will have the rare opportunity of seeing the fruition of their dreams, knowing full well how sound their concepts were.

The present program will be carried out under the direction of John M. Kyle, Chief Engineer of the Port Authority, with the author as Engineer of Design. Mr. O. H. Ammann, who was in charge of the original design of the George Washington Bridge, has been retained as consultant for the bridge modifications.

REFERENCE

1. Journal of the Operations Research Society of America, Vol. 2, No. 2, May, 1954.

Paper 1633 May, 1958 ST 3

Journal of the
STRUCTURAL DIVISION
Proceedings of the American Society of Civil Engineers

DEFLECTION LIMITATIONS OF BRIDGES

Progress Report of the Committee on Deflection Limitations
of Bridges of the Structural Division
(Proc. Paper 1633)

Assignment

The assignment of this committee is to investigate the soundness (rationality) and desirability of Articles 3.6.10 and 3.6.11 of the Standard Specifications for Highway Bridges of the American Association of State Highway Officials. These articles in the 1953 Specifications are:

"3.6.10. – Deflection.

The term "deflection" as used herein shall be the deflection computed in accordance with the assumptions made for loading when computing the stress in the member.

Steel beams or girders having simple or continuous spans shall be designed so that the deflection due to live load plus impact shall not exceed 1/800 of the span, the span length being considered the distance center to center of bearings.

The deflection of cantilever arms due to live load plus impact shall be limited to 1/300 of the cantilever arm.

When bridges have cross bracing or diaphragms sufficient in depth and strength to insure lateral distribution of loads, the deflection may be computed for the standard loading, considering all beams or girders as acting together and having equal deflection.

Sidewalk live load may be neglected in computing deflection.

The moment of inertia of the gross cross-sectional area shall be used for computing the deflections of beams and girders.

The gross area of each member shall be used in computing the deflections of trusses. When perforated cover plates are used in truss members, the effective area shall be the net volume divided by the length from center to center of perforations."

Note: Discussion open until October 1, 1958. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. Paper 1633 is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 84, No. ST 3, May, 1958.

"3.6.11. — Depth Ratios.

The ratio of the depth of the length of spans, preferably shall be not less than the following:

For trusses 1/10
For plate girders and rolled beams used as girders 1/25
For continuous spans, the span length shall be considered as the distance between dead load points of contraflexure.

If depths less than these are used, the sections shall be so increased that the maximum deflection will be not greater than if these ratios had not been exceeded."

Directly involved in the application of Article 3.6.10 are those articles specifying the loading and portions of Section 3.9 concerning composite beams. Less directly related are provisions concerning impact and secondary stresses and certain details of design.

Work of the Committee

In its assignment, the Committee has considered the following:

1. What is the purpose of the two paragraphs, or of other similar requirements?
2. Are the two paragraphs consistent with each other and with other paragraphs of the A.A.S.H.O. specifications?
3. In view of present knowledge, what modifications, if any, are desirable in the two paragraphs?
4. What additional investigations are desirable and to what extent are they economically justified in order to provide a basis for modifying the present specifications?

In regard to the above considerations, the Committee has:

1. Reviewed the history of the deflection limitations for both highway and railway bridges and, since they are intimately related, the corresponding history of loadings, design, materials, and unit stresses.
2. Sent out questionnaires to secure data on the characteristics of bridges subject to "objectionable" vibrations, with special emphasis on what constitutes objectionable vibrations.
3. Reviewed results of tests on highway bridges subjected to dynamic loading.
4. Reviewed data bearing on permissible vibrations, from both structural and psychological aspects.

Although the Committee has confined its activities primarily to steel highway bridges, some of its findings may be applicable to reinforced concrete structures and possibly also to railway bridges.

History

Deflection Limitations

As early as 1871 the specifications of the Phoenix Bridge Company limited the passage of a train and locomotive at 30 miles per hour to 1/1200 of the span.

In 1905 the A.R.E.A. Specifications provided that "pony trusses and plate girders shall preferably have a depth not less than 1/10 of the span and rolled beams and channels used as girders shall preferably have a depth not less than 1/12 of the span. When these ratios are decreased, proper increases shall be made to the flange section."

The published discussion of the 1905 A.R.E.A. specification showed general acceptance of the limiting ratios although the committee records indicate uncertainty as to any basis for a particular value and show a disposition on the part of the committee to leave latitude for the exercise of individual judgment. The limiting ratios have been altered from time to time, as follows:

	Trusses	Plate Girders	Rolled Beams
1905	1/10	1/10	1/12
1907, 1911, 1915	1/10	1/12	1/12
1919, 1921, 1950, 1953	1/10	1/12	1/15

Early highway bridge specifications followed, with minor modifications, the lead of the railroads regarding depth limitations. The U. S. Department of Agriculture Circular No. 100, issued in 1913 by the Bureau of Public Roads, set limiting depth ratios of 1/10, 1/12, and 1/20 of the span for trusses, plate girders, and rolled beams, respectively. U. S. Department of Agriculture Bulletin No. 1259, issued in October, 1924 (A.A.S.H.O. Specifications), contained the same limitations. The following table illustrates the transition in these requirements:

	Trusses	Plate Girders	Rolled Beams
1913, 1924	1/10	1/12	1/20
1931	1/10	1/15	1/20
1935, 1941, 1949, 1953	1/10	1/25	

In 1935 the following paragraph was added:

"If depths less than these are used, the sections shall be so increased that the maximum deflection will be not greater than if these ratios had not been exceeded."

In the early 1930's, reports of the vibration of steel highway girder bridges began to appear. These often involved bridges consisting of continuous spans or cantilevers with suspended spans which were then coming into general use. The Bureau of Public Roads made a statistical study in an attempt to correlate the reported vibrations with the properties of the bridges involved. This study suggested that the bridges for which objectionable vibrations had been reported were usually structures for which the computed live load deflections were more than 1/800 of the span for simple and continuous spans and more than 1/400 for cantilever arms. Accordingly, the design specifications issued

by the Bureau of Public Roads in November 1936 limited the computed deflection under live load plus impact to 1/800 of the span for bridges carrying light traffic, and 1/1100 of the span for bridges in or adjacent to populous centers and bridges carrying heavy traffic. In 1939 the specification was extended to limit the deflection for cantilever arms to 1/300 of the arm, "except at locations of dense traffic where the length of cantilever arm divided by 400 shall be used." These deflection limitations were discussed and adopted in meetings of the A.A.S.H.O. Bridge Committee in 1938. The present requirements were first used in the 1941 specification. However, certain state highway departments limit the live-load-plus impact deflection to 1/1100 of the span.

General Design

It is pertinent to note the trend in highway bridge design during the period in which this development of deflection and depth limitations occurred. When the first limitations were proposed, the standard floor was plank, the supporting members were either simple beams, pony trusses or pin-connected through trusses, and questions concerning the effects of vibration centered on individual members. These questions were largely eliminated with the advent of more substantial members, riveted joints, and concrete floors. Vibrations of the bridge as a whole were noted with some concern at about the time unit stresses were increased and cantilever and continuous construction began to appear. Today the trend to heavier live loading, higher stresses, welded construction, composite design, and the increasing use of light continuous construction is causing concern as to the limitations adopted in the early 1930's.

Loadings

At the turn of the century, the heaviest loading was a road roller. With the development of the motor truck, design loadings were first based on 10-ton and 15-ton trucks. The 20-ton truck loading was introduced in 1924 for bridges on which heavy traffic was anticipated during the life of the structure. In the 1927 "Specifications for Steel Highway Bridges," the Conference Committee representing the A.A.S.H.O. and the A.R.E.A. introduced the truck train, a heavy truck preceded and followed by trucks having three-fourths of its weight. This was superseded in 1941 by the equivalent lane loading, which was first used as an optional loading in the 1931 A.A.S.H.O. Bridge Specifications. The H20-S16 loading was introduced in the 1941 A.A.S.H.O. Specification.

It is significant to note that the 20-ton truck loading was used in 1924 to provide, not for current loads, but for possible increases within the life of the bridge, whereas the present H20-S16 vehicle represents quite closely the present legal and actual loading in many states, except that dual axles are assumed for the two 32,000 lb. loads, a condition which has slight effect on the design of usual beam spans.

The impact allowance, combined with live load in determining limiting deflections, is that fraction of the live load stress represented by the formula: $I = \frac{50}{L + 125}$, with a maximum value of 30 per cent. It was first used in the Conference Committee Specification of 1927 and has developed from a dual background. Its general form is that of a group of curves drawn to represent, with varying conservatism, the envelope of the results of early tests on

railroad bridges showing the per cent of impact as a function of span. The general level and limiting value of the formula show the influence of A. H. Fuller's impact tests on highway bridges in Iowa in 1922-25.

Materials and Unit Stresses

In respect to deflection limitations, the principal significance of the evolution of materials and unit stresses, and to some extent of loading as well, lies in their effect upon the economic depth-to-span ratios. While early committees found it difficult to decide upon an optimum depth-span ratio from the standpoint of deflection and vibration, they could compute such a ratio on the basis of economy. This economic ratio gradually changed from 1/7 to 1/12 during the close of the 19th century. It seems likely, therefore, that the early limitations on depth-span ratio were primarily economic in origin. The advantages offered by the development of larger rolled beams are reflected in the gradual decrease in limiting depth-span ratio. The change has been less in railway than in highway bridge design because of the more limited span for which rolled sections can carry the heavier railroad loads. Recently a concession was made to further the use of composite steel and concrete construction at the time the A.A.S.H.O. Committee on Bridges and Structures applied the specified depth-span ratio of 1/25 to the combined depth of the beam and the slab, adding a limit of 1/30 for the beam alone.

The recent trend toward composite construction for steel girder bridges raises a further question regarding the continued effectiveness of a limitation on computed live load deflection. Field tests have shown that bridges designed without composite action may actually exhibit a considerable degree of composite action under traffic as a result of bond or friction between the girders and slab. For this reason, the actual live-load deflection of such bridges may be significantly less than that computed neglecting composite action. On the other hand, the actual deflection of bridges designed for composite action will be relatively close to the computed deflection. Calculations for typical simple-span I-beam bridges show that bridges designed for composite action may have live-load deflections ranging from 9 to 54 per cent greater than bridges of non-composite design, if it is assumed that composite action exists under service conditions in both cases. For all but the shorter spans, the increase in deflection is generally about 25 per cent. Thus, a deflection limitation of 1/1000 of the span for composite designs would be necessary in order to provide approximately the same stiffness as the present limit of 1/800 provides for non-composite designs.

Purpose of Limitations

Historically it appears that the primary purpose of limiting depth-span ratios was to limit the live load deflection. The ratios adopted, however, show the influence of economic depth under changing specifications for load and unit stress. In the case of railroad bridges, deflection was limited to avoid excessive vibration of the structure in resonance with the recurring hammer blows of the locomotive drive wheels and also to avoid objectionable oscillation of the rolling stock induced when the deflections of successive spans tended to set up a harmonic excitation of the sprung weight.

Early highway bridge specifications followed the lead of those for railroad bridges. It was not until the early 1930's that a distinctive limitation tailored

to a distinctive problem of highway bridge design was developed.

In summarizing the purposes of deflection limitations for highway bridges, it should first be noted that the safety of the structure is not involved, even to the extent that more flexible bridges may be adequate to carry the live load with safety.

The following reasons for limiting deflections have been considered by the Committee:

1. To avoid undesirable structural effects, including:
   a) Excessive deformation stresses in secondary members or connections resulting either from the deflection itself or from induced rotations at joints or supports.
   b) Excessive dynamic stresses of the type considered in design by the use of conventional "impact" factors.
   c) Fatigue effects resulting from excessive vibration.
2. To avoid undesirable psychological reactions by:
   a) Pedestrians, whose reactions are clearly a consequence of the motion of the bridge alone.
   b) Passengers in vehicles, whose reactions are affected by the resultant motion of the vehicle in combination with the bridge or by the motion of the bridge when the vehicle is at rest on the span.

Doubts concerning the applicability of the specifications arise from two general considerations in their relation to the above objectives:

First, it is difficult to determine how much vibration of flexibility is acceptable. Fatigue and structural damage to details may result from large amplitudes; objectionable riding qualities may be observed at smaller amplitudes, and pedestrians may require even greater stiffness for their reassurance. What limit on amplitude is dictated by each factor and to what extent are these limits affected by the frequency of vibration? Can structural damage be avoided more economically by suitable details of design than by increased stiffness? Can the tolerance to vibration of pedestrians and passengers of vehicles be raised?

Second, the objectionable features of vibration of a bridge are not entirely controlled by a fixed limitation of deflection based on live loading. The optimum deflection limitation as it affects vibration varies with the type of structure, relative weight of structure and vehicles, and the speed and density of traffic. Geometric design, details, and imperfections in the riding surface may also have a strong influence. And finally, although increased stiffness may reduce the amplitude of or the tendency to resonant vibration, it actually increases the dynamic stress caused by individual blows.

Investigations

Survey

The committee sent out a questionnaire to state highway departments and other organizations experienced in designing and maintaining highway bridges in order to assemble data on the behavior of bridges in relation to their design characteristics, to learn, if possible, the optimum limitation of deflection, and to obtain the views of experienced engineers. As expected, opinion differed. The replies to this questionnaire may be summarized as follows:

1. The majority of those replying felt that the present limitations are proper and that no changes should be made without valid reasons. Some, especially those in metropolitan areas, expressed the belief that the restrictions should be tightened.
2. With the exception of a few cases of failures in stringer connections and cracking of concrete floors at points of negative moment, there were no reports of structural damage.
3. It was frequently observed that maximum apparent oscillation occurred with the passage of medium weight rather than heavier vehicles.
4. In proportion to their number, there were more reports of objectionable vibrations with continuous than with simple spans.
5. In general, there were no definite statements as to what constitutes undesirable vibrations. The principal reason for greater rigidity seemed to be to avoid undesirable pedestrian reaction.

It was evident from the results of the survey that an engineer's advocacy of stiffer or more flexible bridges could be related to his experience with metropolitan traffic on the one hand or with predominantly rural traffic on the other. This fact does not necessarily indicate a need for different stiffness requirements for urban or truck lines than for rural or light traffic roads; more likely, it indicates only the less frequent use by pedestrians of bridges in rural areas.

Tests

The problem of impact and vibration has been investigated during the past generation and earlier. A. H. Fuller's tests on several bridges in Iowa in 1922-25 (21,23,95) were among the early comprehensive studies of highway bridges. There have been numerous observations by individuals concerned about the vibration of bridges, but often their instruments were simple and their investigations yielded only qualitative results. Others were more quantitative.

In recent years investigators have made increasing use of the newer instruments to obtain quantitative measurements of static and dynamic deflection and strain, frequency, structural damping, and the lateral distribution of strain and deflection. Field tests of this type on actual highway bridges have been made in recent years in Ohio,(66) New Zealand,(90) Pennsylvania,(94) Michigan,(96,106,107) California,(97) Minnesota,(100) Massachusetts,(101,108) Oregon,(102) and Indiana.(105) Additional field tests were made in Missouri and South Dakota in 1955, in Iowa and Nebraska in 1956, and in Ontario in 1956 and 1957, but the results have not yet been published.

Analyses of vibration or impact phenomena have been made in connection with most of the field tests mentioned above as an aid to interpreting the results obtained. In addition, more extensive and more general analytical studies of the impact problem have been carried out at the Univeristy of California,(103) at the University of Illinois,(104) and at the Massachusetts Institute of Technology.(108) Investigations at the last two institutions have also included laboratory tests.

Dynamic Action

The stress produced by a vehicle moving rapidly across a highway bridge may be significantly greater than the stress which would be produced by the same load moving very slowly or standing still. This increase of dynamic stress over static stress is commonly referred to as the impact effect and is considered in all contemporary specifications for the design of highway bridges.

The dynamic effect of a moving load results entirely from the fact that the masses of the bridge and vehicle are accelerated and the forces exerted between them are thus different from the "static" forced due solely to the acceleration of gravity. As a load moves across a bridge span, the bridge deflects. Since this deflection occurs relatively rapidly, the mass of the bridge is accelerated and the bridge is thus subjected to inertia forces in addition to the static force exerted by the moving load. If the moving load had no mass, the dynamic effect would be due solely to the added inertia force on the bridge resulting from the acceleration of the mass of the bridge itself. However, since a vehicle does have mass, the force exerted by the vehicle may be greater or less than the weight of the vehicle because of the inertia forces due to the acceleration of the mass of the vehicle. Moreover, because of the flexibility of the vehicle springs and tires, the vertical motion and acceleration of the vehicle usually are not the same as those of the bridge even if there is no "bounce" or separation.

The forces which produce stresses in a bridge under dynamic loading are the result of combining the following three quantities:

a) The static weight of the vehicle.
b) The inertia forces due to vertical acceleration of the mass of the vehicle.
c) The inertia forces due to vertical acceleration of the mass of the bridge.

The motions and consequent accelerations of the vehicle and of the bridge are not independent of each other and are affected by some or all of the following factors:

a) The flexibility of the bridge, and its natural frequency of vibration.
b) The flexibility of the vehicle springs and tires, and the natural frequencies of vehicle vibration.
c) The relative weights of the vehicle or vehicles and the bridge.
d) The speed of the vehicle.
e) The profile of the approach roadway and of the bridge deck itself.
f) The frequency of load application from the passage of successive axles of one or more vehicles.
g) The motion induced by loads in adjacent spans of continuous bridges.
h) The damping characteristics of the bridge and vehicle.

Considering first the case of a smoothly rolling load, the dynamic increment of bridge deflection results from accelerations which cannot be maintained indefinitely as can a static load. Therefore, the elastic resistance of the bridge acts as a restoring force which tends to return the bridge to its static deflected position after the maximum dynamic deflection is reached. However, since the upward movement of the bridge also involves accelerations, the deflection decreases to a value less than the static deflection. As this pattern is repeated, during the passage of a load, the bridge oscillates at a

frequency corresponding approximately to the natural frequency of the loaded bridge. Because of the varying position of the moving vehicle, this frequency varies during the passage of the load, being somewhat below the natural frequency of the bridge alone unless some other frequency is forced by the unbalanced action of the vehicle. Simple spans generally oscillate in the fundamental mode while continuous spans may oscillate in the first symmetric or first anti-symmetric mode. Higher frequency modes of small amplitude are usually superimposed on the dominant vibration. The residual vibration continuing after the load leaves the bridge is at the natural frequency of the unloaded structure.

The frequency of vibration observed in tests frequently checks the value computed on the basis of full composite action, whether or not shear connectors were provided in the design. However, in some cases, the observed frequencies for bridges of non-composite design are checked more closely by assuming only 50 per cent composite action.

Since the dynamic increment of deflection is oscillatory in nature, the maximum total effect as related to either deflection or stress may be greater or less than the maximum static effect. For a given bridge and vehicle, the maximum dynamic effect will occur at a certain vehicle speed for which the bridge oscillations combine with the static effects in the most unfavorable manner. At other speeds, the resultant stresses and deflections will be less than this maximum and may even be less than those produced by a static load. In general, the maximum dynamic stress will occur near the point on the span at which the static stress would be a maximum; that is, near midspan. If the motion of the vehicle is neglected, the maximum dynamic effect will usually occur beyond midspan in the direction of the movement of the vehicle; however, if the mass of the vehicle is also oscillating, the maximum effect may occur at almost any location near midspan.

The dynamic effects due to motions of the bridge or vehicle may be greatly accentuated if conditions are such as to make the forces acting on the bridge periodic in nature and if the period of these forces is close or equal to the natural period of vibration of the bridge or vehicle. Such periodic forces may be produced by:

a) The passage of more than one heavy axle or more than one vehicle at a particular speed and spacing.
b) The presence of an unbalanced wheel or of a high spot on a vehicle tire.
c) The existence of a systematic variation in the profile of the bridge deck.
d) Surface roughness of the approaches to the bridge, which may induce an oscillation of the vehicle mass, as it enters the span.

Although few data are available, it is evident that damping cannot be neglected, especially in connection with the motions of the vehicle itself. In particular, the effects of periodic forces will be reduced by damping either in the springs of the vehicle or in the bridge itself. However, in the case of oscillations due to springing in the tires alone, damping may be less significant.

The effect of a systematic variation in the profile of the bridge deck has been mentioned as a possible cause of increased dynamic effects. A localized discontinuity in the bridge profile, such as a bump or depression, produces effects of a different type. The principal effect of an individual wheel striking a single obstruction is to produce a blow or true impact. The energy of this blow is absorbed by an added force on the bridge moving through a distance permitted by the resulting deflection of the bridge, the vehicle springs, and

the vehicle tire. Thus, a stiff bridge may be expected to receive a higher impact force from the same blow than would a more flexible one. However, the action is complicated; the wheel striking the obstruction first exerts an added force, then partially or completely separates from the bridge, and then falls upon it. While the load is removed, the bridge recovers to the extent possible during the brief period of relief from load. The return blow produced an oscillation at about the natural frequency of the bridge, which, however, diminishes with time as a function of the damping of the structure if not maintained by subsequent effects from the moving vehicle. The blow may be timed to augment or decrease any previously excited oscillations.

Since the force applied to the bridge is a function of the vertical motion of the vehicle on its springs and tires, the effect of approach roadway roughness as it affects the oscillation of the vehicle cannot be neglected. If the vehicle mass is already oscillating when it enters the bridge, the resulting dynamic effect will depend on how this oscillation adds to or subtracts from that produced by the bridge motion itself and how the oscillations of the vehicle and bridge are combined near the location of maximum static effects. Again, however, the effects of damping in the vehicle spring system may be significant.

The foregoing discussion is generally applicable to the case of a single load or line of loads moving across a single beam. Where more than one beam is involved, the lateral distribution of the bridge response in terms of deflection or stress can affect the magnitude of the maximum dynamic effects. Where the load is spread laterally, as on the two wheels of a single axle, the forces transmitted to the individual beams are affected by the "roll," or lateral oscillation of the vehicle on its springs and tires, which may vary greatly since a sharp turn or lurch may transfer nearly all of the weight of the vehicle to the wheels on one side.

Tests of actual bridges have shown a wide variation in the relative influence of these several factors upon the amplitudes of the strain and deflection oscillations. Some tests have shown sharp increases in the amplitude at certain speeds which could be correlated with the axle spacings and attributed to synchronized effects of the several axle loads. Other tests have shown no such correlation with axle spacing or vehicle speed, but have shown that the principal factor was the vertical oscillation of the vehicle on its springs and tires. Another test of two bridges built to the same design but with material differences in the profile of the bridge deck, showed that a roughly systematic variation in the profile of one of the bridges was the dominant factor affecting the amplitude of the dynamic effects.

Psychological Reaction to Bridge Vibration

As mentioned previously, the present limitations on highway bridge deflection in terms of span length were imposed in response to reports of objectionable vibrations. Nevertheless, the results of the survey made by the committee indicate that such reports are still prevalent and tend to come more frequently from pedestrians than from passengers in vehicles.

On the basis of the limited evidence available, it seems reasonable to conclude that the unfavorable reactions which have been reported are related to the dynamics of bridge vibration and not to the static deflections which are controlled by the existing limitations. Moreover, it has not been possible to establish a correlation between "objectionable" vibrations, from a

physiological point of view, and the magnitude of static deflection.

Much more needs to be known about the actual magnitudes and other characteristics of bridge vibrations which have been reported as "objectionable." Although the human perception of vibratory motion is subjective, and thus not susceptible to direct quantitative measurement, a great deal of effort has been given to correlations of human sensations with known magnitudes of vibration.

Sensitivity seems to be related to different vibratory characteristics—velocity, acceleration, and rate of change of acceleration—at different frequencies. As might be expected, there is some lack of agreement between various proposed scales as to just what vibration intensities could be termed "barely perceptible," "perceptible," "annoying" and "unpleasant." This is perhaps not unreasonable when it is considered that human reactions must represent a spectrum of effects of various degrees of severity, which are further influenced by such things as environment and the duration of the sensation.

In a study of the assessment of vibrations in buildings, in which most of the proposed scales of sensitivity are reviewed,[98] it is suggested that the Reiher-Meister scale[37] is best suited to vibrations in building structures. More recently, additional scales of perceptivity have been proposed.[77] There is no evidence to indicate which, if any, of the proposed scales are suited to the description of human reactions to bridge vibrations. Only by comparing actual motions with the various scales could any criterion of objectionable vibrations in bridges be established.

Although the results of these and other studies[40,44,63,87] may not necessarily be applicable to the problem of bridge vibrations, they serve to emphasize the dependence of psychological reactions on dynamic factors such as frequency, acceleration, and rate of change of acceleration rather than on amplitude alone. Nevertheless, the control of static deflection has some influence on the dynamic characteristics of bridges and further studies are indicated to determine whether deflection limitations are by themselves sufficient to eliminate objectionable vibrations.

Also, it is not clear whether unfavorable physiological reactions result from actual discomfort as a result of the bridge vibration or simply from the unexpected motion of a structure which the layman might logically expect to be rigid. If the latter situation exists to any great degree, a program of "consumer education" might prove to be helpful in reducing complaints of objectionable vibrations. In some cases, the number of such complaints has been reduced eventually simply because the users of a particular bridge gradually became accustomed to its behavior.

SUMMARY

The A.A.S.H.O. Specifications which the Committee was assigned to study are probably unique in requiring a dual control on deflection in the form of limiting depth-span ratios and deflection-span ratios.

The limitation on depth in terms of span presumably had its origin in specifications for railroad bridges. It is at least partly economic in origin, but its use to control deflection is clearly indicated by the nature of the provisions for designs of shallower depth. The limiting depth-span ratio has changed through the years, for both railroad and highway bridges, and has become

more and more liberal for plate girders and rolled beams in spite of increases in working stresses. Neither the reasons for these changes nor the original basis for the limitation on depth-span ratios have been determined by the Committee. In 1905, the A.R.E.A. Committee explained the somewhat ambiguous wording of their provisions for design which reduced depths as follows:

> "We established the rule because we could not agree on any. Some of us in designing a girder that is very shallow in proportion to its length, decrease the unit stress or increase the section according to some rule which we guess at. We put that in there so that a man would have a warrant for using whatever he pleased."

The limitation on deflection in terms of span was inspired by complaints of objectionable vibrations in existing bridges, and was based on a limited statistical study involving comparisons of computed deflections.

Since these two criteria have different origins and possibly even different objectives, it is difficult to determine their soundness or desirability in the light of the extremely limited data available. The significant effect of depth on the relative deflections of two beams designed for the same unit stress suggests the value of a limiting depth-span ratio, especially since this criterion can be applied directly without the need for trial designs. However, since controlling the depth will control only the total deflection, this limit becomes ineffective for high live load-dead load ratios. The direct limitation on deflection will control usually only for heavy live loads but even here its soundness depends directly on the suitable choice of limiting deflections in terms of either psychological reaction or structural damage.

The limited survey conducted by the Committee revealed no evidence of serious structural damage that could be attributed to excessive deflection. The few examples of damaged stringer connections or cracked concrete floors could probably be corrected more effectively by changes in design than by more restrictive limitations on deflection. On the other hand, both the historical study and the results from the survey indicate clearly that unfavorable psychological reaction to bridge deflection or vibration is probably the most frequent and important source of concern regarding the flexibility of bridges. However, those characteristics of bridge vibration which are considered objectionable by pedestrians or passengers in vehicles cannot yet be defined.

The Committee has been forced to conclude, therefore, that the information now available is not sufficient to warrant any recommendations regarding either what constitutes objectionable deflection or vibration or how best to limit them.

RECOMMENDATIONS

1. It is recommended that no changes be made at this time in the present requirements regarding the limitation of depth-span or deflection-span ratios for highway bridges. Although the Committee has discussed minor modifications of the present specifications, such as a more realistic method of computing live load deflection, the essentially empirical nature of the present limitations does not lend itself to such revisions no matter how logical they may seem. It is the belief of the Committee that the present limitations are

about as good as any that can be set up until more factual information is available. Granted this, any modification of the method of computing deflection should be accompanied by a corresponding temporary modification of the limiting ratios in order to maintain the present level of actual minimum flexibility pending a more complete investigation of the entire subject. However, the Committee recommends the use of a somewhat more restrictive deflection limit for bridges in which composite action is taken into account in design.

2. It is recommended that further attempts be made to determine what constitutes objectionable deflection or vibration of highway bridges and to develop design criteria which will limit them. The phenomena to be studied should include the following:

a) Structural damage due to excessive deformation stresses. The degree to which such damage, if any, can be avoided by changes in design should be determined.
b) Unfavorable psychological reactions from bridge users. An effort should be made to determine which characteristics of bridge vibrations cause the unfavorable reactions and, if possible, to describe these characteristics in engineering terms. Further study is needed to determine whether unfavorable reactions by bridge users are serious enough to warrant changes in bridge design practices or whether a program of education might be a satisfactory solution to this aspect of the problem.
c) Dynamic or impact stresses. Since the impact effect is directly related to bridge flexibility, no changes should be made in the present limitations on depth or deflection without considering their possible effects on the proper allowance for dynamic stresses in design. The Committee believes that this aspect of the bridge deflection problem is the most important of those discussed in this report and that further research on this subject should be encouraged as a necessary condition to the improvement of present design procedures. Only when the dynamic behavior of bridges under moving loads is fully understood will it be possible to consider deflection or vibration in a rational manner.

Respectfully submitted,

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Paper 1634 May, 1958 ST 3

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RIGID FRAME ANALYSIS WITH THE AID OF DIGITAL COMPUTERS

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(Proc. Paper 1634)

SYNOPSIS

This paper shows how complicated statically indeterminate structures may be analyzed, with ease, through existing mathematical routines and basic engineering functional relationships. The method is explained through illustrative examples which include a multibay gabled bent and lean-to frames with tapered girders. The results, which may include in addition to the restraining moments of all the members, the reactions, column deflections and joint rotations, are obtained simultaneously for the various loading conditions required, including the effects of foundation settlement, shrinkage, temperature changes and various other factors of refinement and special features of framing that the structural engineer may wish to consider. Most of the nomenclature will be recognized as that commonly used with the moment distribution and slope deflection methods of frame analysis.

INTRODUCTION

The analysis of highly indeterminate structures can at best be said to be a complicated and difficult matter. Much of the drudgery and difficulty have been due to the need of solving simultaneous equations. Obviously, methods were desired that would reduce the number of equations to be solved. Numerous short-cuts and specialized solutions were presented, all in an effort to reduce to a minimum the number of unknowns necessary for their solution.

A considerable reduction in the number of equations was accomplished with the slope-deflection method, in which deformations, instead of the redundant forces are used as the unknowns. The (slope-deflection) equations for the restraining moments are written in terms of the end rotations and the relative

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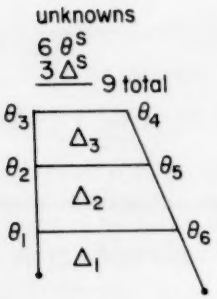


FIG. 1

displacements of the end supports. The numerical values of these deformations are calculated from the necessary equilibrium conditions, and then substituted into previously formulated moment equations to obtain the values of the redundant moments necessary for design. The horizontal and vertical reactions may then be calculated from the equations of static equilibrium for the various components of the structure (free body). Many systematic procedures were developed in the past, that may be used successfully in solving simultaneous equations, with a limited number of unknowns. The time required, to solve the equations, increases rapidly with the number of unknowns, and for large systems the labor involved becomes prohibitive. A problem in structural analysis involving the solution of, say, nine simultaneous equations (Fig. 1) might require almost a day's computation.

With the introduction of the Cross method of moment distribution the number of simultaneous equations was further reduced to the number of degrees of freedom of motion, that is, the number of independent joint translations. The deflection pattern of the structure in Fig. 1 is defined completely by the 3-story displacements, and thus three simultaneous equations are required. Essentially, the moment distribution method consists of solving the simultaneous equations in the slope-deflection method by successive approximations. Convergence is rapid only in the case of joint rotations, when translatory motion is either absent or temporarily prevented. When a joint is rotated to equilibrium it exerts primarily local effects, whereas, when same joint undergoes motion of translation it may affect considerably the restraining moments of many members. To control the translation of the joints, the auxiliary force systems are expressed through independent equilibrium equations, one for each needed deflection. In the two-bay gabled bent of example 1, the deflection pattern is defined completely by specifying the three deflections of the column tops. The analysis of frames has further been made tedious, by the time consuming necessity of repeating the entire procedure for each new loading condition, or for each new trial section to achieve an economical design. The drawback of such methods, that yield numerical solutions to only one problem at a time, are especially apparent when the structure contains members with variable cross-section. The increasing popularity of plastic analysis in steel design may be largely attributed to "simplicity" in determining the maximum load the structure will support (as determined from an analysis of strength in the plastic range). Plastic analysis of multistory frames is all but simple.

Due to the intricacy of many a solution, and the ever-lasting search for the

short-cut, and for the approximate solution that gives almost "exact" results, there is an increasing tendency on the part of design engineers to confuse the terms "mathematical" with "theoretical" and "exact." It must be clearly understood, that mathematics, as far as structural engineering is concerned, is merely a tool for practical use, not an end in itself. No matter how intricate and lengthy the calculations may be, the results are not any better than the assumptions upon which the mathematical theory and the physical data are based. It applies equally well to whether the engineer uses his slide rule, a desk calculator, or a giant electronic digital computer. The correct solution of the equations provides assurance of mathematical accuracy only. However, the uncertainty of certain assumptions does not justify the substitution of crude guesses for sensible analysis and correct calculations. The electronic digital computer cannot improve the engineer's assumptions, but can perform all the necessary arithmetic for him and free him to do engineering in the true and creative sense. Together with a mathematical routine library, it becomes a highly trained specialist that can convert complete mathematical operations into basic arithmetic elements, and perform all the necessary computations at tremendous speed. Many general purpose mathematical programs were coded for the various computers and are readily available from the computer manufacturers.

Programs for solving simultaneous linear equations with many unknowns, solutions of systems of differential equations, numerical integration and the various transcendental functions are especially helpful to the structural engineer. He can make effective use of these programs without knowing much about programming himself. However, until the structural engineer learns at least the fundamentals of computer programming, he is not likely to be aware of how to make the computer serve him best. The time is rapidly approaching when the modern design engineer cannot afford any longer to be without some computer know-how, and the procedures involved in obtaining solutions through electronic computation.

Frame Analysis

The distinction between statically determinate and indeterminate structures is essentially a mathematical one. In the case of indeterminate analysis, the equations of static equilibrium are supplemented by equations involving the elastic behavior of the structure. Indeterminate structures, that are dependent for their strength and stability on the capacity of the joints to transmit bending moments with little or no deformation, are known as "rigid-frames." Though such continuity often exists in many directions, the usual procedure is to neglect torsional distortion and to consider the frame action in each direction separately as a two dimensional problem. With the large systems of simultaneous equations that can be solved easily with the aid of a digital computer, the three dimensional analysis of the space structure is no longer the formidable problem it once was. In the examples that follow only conventional coplanar frames are considered. As is common practice in structural engineering the effect of the axial and shear forces on the deformations (and consequently on the moments) was disregarded.

Equations (1) and (2) are the well known slope-deflection equations for a restrained member with constant moments of inertia, due to flexure only.

$$M_{AB} = \frac{2EI}{L}\left(2\theta_A + \theta_B - 3\frac{\Delta}{L}\right) + M_A^F \quad (1)$$

$$M_{BA} = \frac{2EI}{L}\left(\theta_A + 2\theta_B - 3\frac{\Delta}{L}\right) + M_B^F \quad (2)$$

where E, I, and L, are the modulus of elasticity, moment of inertia and length respectively.

M_{AB} = restraining moment at end A

M_{BA} = restraining moment at end B

M_A^F = fixed-end moment at end A

M_B^F = fixed-end moment at end B

now let

$$\frac{4EI}{L} = C_1 \qquad \frac{2EI}{L} = C_2 \qquad \frac{6EI}{L^2} = C_3$$

Equations (1) and (2) for the restrained member A-B may now be rewritten to the more convenient form of (3) and (4), in which all the unknown quantities are accumulated in the left hand side of the equations. The signs of the

$$M_{AB} - C_1\theta_A - C_2\theta_B + C_3\Delta = +M_A^F \quad (3)$$

$$M_{BA} - C_2\theta_A - C_1\theta_B + C_3\Delta = +M_B^F \quad (4)$$

deformations follow the usual convention of considering rotations and displacements of end supports as positive if they produce a clockwise rotation. When writing the equations the sign of the deformations is assumed positive (see Fig. 2). A minus sign, in the results, then indicates that the displacement of the joint is in opposition to the sense assumed.

The coplanar end forces acting upon the restrained member are represented by a horizontal force H, a vertical force V and a moment M. Strict attention must be paid to signs as they frequently prove to be the largest source of errors. From the point of view of equilibrium no essential distinction can be made between active and reactive forces. Thus, a lot of confusion with regard to signs may be avoided if the same sign is adapted to all forces, and no

θ_A = rotation at end A

θ_B = rotation at end B

$\Delta = \delta_B - \delta_A$ = relative displacement of end supports

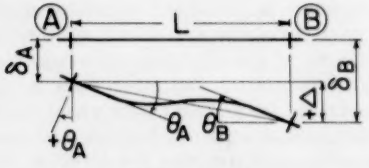


FIG. 2

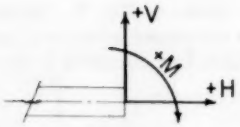


FIG. 3

distinction made between external loads and reactions. Fig. 3 shows the assumed positive direction of the forces, when acting on the member. All vertical loads (or reactions) are taken as positive when acting upward. Horizontal forces are positive when acting to the right, and moments are positive when acting clockwise. At the outset all unknowns are assumed as positive, and are grouped in the left hand side of the equations. The right hand side contains the load function or constant of the equation. For the hypothetical case where all external loading is absent, the structure is unstrained and all the load constants in the equations are zero. The left hand portion of the equations, then, may be termed "the unstrained equations." Since the unstrained equations are dependent only on the characteristics of the structure and remain the same for all loading conditions, it is convenient to write them first. The various loading conditions, including thermal effects and displacement of supports can then be arranged in tabular form to the right of the unstrained equations, and all necessary results obtained simultaneously.

In the case of "ultimate strength design" of reinforced concrete, the present ACI code provides that the ultimate-strength term be confined to design of the cross section only, and that the external moments and forces acting in a structure be determined by the theory of elastic frames. The load factors in section A604 of the code are intended to provide an ample factor of safety against an increase in live load and also to limit the strain under service loads, to avoid excessive cracking. These criteria are satisfied by the given six expressions. Though it is frequently possible to establish which ones govern specific cases, the advantage of a method which enables the designer to consider all the necessary conditions in one step is inherently obvious. The following examples will illustrate the method.

Example 1

A two-bay unsymmetrical gabled frame is shown in Fig. 4. The relative stiffnesses as well as the assumed deflections, are indicated in the figure. It is required to determine the restraining moments at the end of each member,

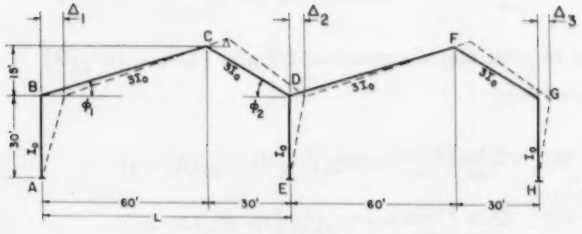


FIG. 4

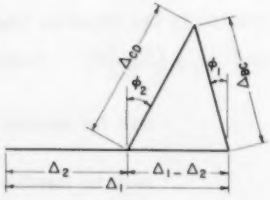


FIG. 5

column reactions, rotations for all joints and deflections at column tops, for the various loading conditions shown in Fig. 7. Obtain results for, both, fixed and hinged column bases. For the cases involving foundation displacement and temperature changes, assume I of columns to be $100(\text{in})^4$. Use $E = 30{,}000{,}000$ psi.

Number of equations

It is obvious that as many equations are needed as there are unknowns. The frame has 8 joints connected by 7 members. For each joint there will be one unknown rotation, and for each member 2 unknown moments. For each column there is one unknown deflection. The 3 vertical and 3 horizontal components of the column reactions add 6 unknowns, with a sub-total of 31. For the particular case, when the column bases are assumed fixed, the rotations at A, E & H are zero, and for the case of hinged bases, the moments at the column base are zero. Thus, compatability with the assumed restraint provides one zero solution for each column, reducing to 28 the number of equations that are required to solve for all the unknowns without subsequent substitution. However, introducing additional unknowns may, frequently, simplify several of the equilibrium equations. In considering the equilibrium of joints, where two sloping members meet, the additional unknowns may be conveniently taken as the horizontal and vertical components of the shears at their ends. Adding the eight unknowns at joints C and F results in a total of 36 simultaneous equations.

Joint displacement

The lateral displacements of joints B, D and G are necessary and sufficient to determine the deflection pattern of the structure.

Let the displacement of joint B be Δ_1 to the right
displacement of joint D be Δ_2 to the right
displacement of joint G be Δ_3 to the right
also assuming that $\Delta_1 > \Delta_2 > \Delta_3$ and ignoring axial deformation
displacement of joint C will be $(\Delta_1 - \Delta_2)$
displacement of joint F will be $(\Delta_2 - \Delta_3)$

The rotation of the sloping members may be obtained from the simple geometric relations of the displacement diagram, Fig. 5.

$$\text{rotation of BC} = -\frac{\Delta_{BC}}{L_{BC}} = -\frac{(\Delta_1-\Delta_2)}{L}\cot\phi_2 = -(\Delta_1-\Delta_2)\frac{2}{90}$$

$$\text{rotation of CD} = +\frac{\Delta_{CD}}{L_{CD}} = +\frac{(\Delta_1-\Delta_2)}{L}\cot\phi_1 = +(\Delta_1-\Delta_2)\frac{4}{90}$$

Similarly the rotation (due to joint displacement) of DF and FG is $-\left(\frac{\Delta_2-\Delta_3}{45}\right)$ and $+\left(\frac{\Delta_2-\Delta_3}{22.5}\right)$ respectively.

$$\text{also, for member BC} \quad \frac{2EI_{BC}}{L_{BC}}\left(\frac{3\Delta_{BC}}{L_{BC}}\right) = -3C_2\left(\frac{\Delta_1-\Delta_2}{45}\right) = C'_{BC}(\Delta_1-\Delta_2)$$

$$\text{and, for member CD} \quad \frac{2EI_{CD}}{L_{CD}}\left(\frac{3\Delta_{CD}}{L_{CD}}\right) = +3C_2\left(\frac{\Delta_1-\Delta_2}{22.5}\right) = C'_{CD}(\Delta_1-\Delta_2)$$

The numerical values of the coefficients needed for the slope-deflection equations are tabulated below.

Members	$L_{(FT)}$	EI_0 (relative values)	C_1	C_2	C_3	C'
AB, ED, HG	30.00	100	13.33	6.67	0.667	—
BC, DF	61.85	300	19.40	9.70	0.470	− 0.647
CD, FG	33.54	300	35.78	17.89	1.600	+ 2.385

The slope-deflection and necessary equilibrium equations for the fifteen loading conditions considered are shown in Fig. 7. Next the coefficients of the unknowns and the constants are arranged in matrix form, Table III and the mathematical program can then be applied to obtain the solutions to the equations. When fixed column bases are considered the columns containing the base rotations are simply crossed out, while for hinged bases, the reaction moments are crossed out and the rotations retained. The blank spaces in Table III indicate zeros. The results are tabulated in Table I for the case of fixed bases and Table II for column bases assumed hinged. Two digits past the decimal point were retained for shears, moments and deflections, while rotations are given to three places past the decimal. The results are also shown in Fig. 8. The values in parentheses are for the case of hinged bases. It should be noted that since relative rigidities were used, the deformation values in the results should be multiplied by a proper dimensional constant if

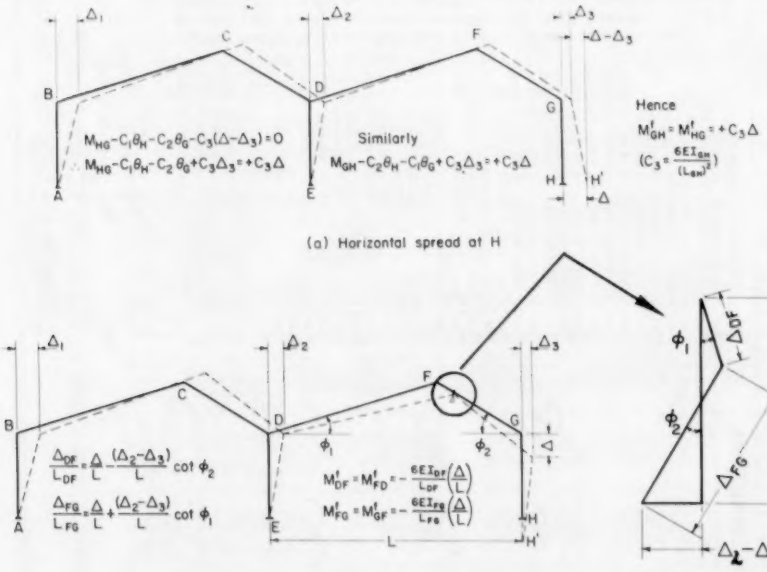
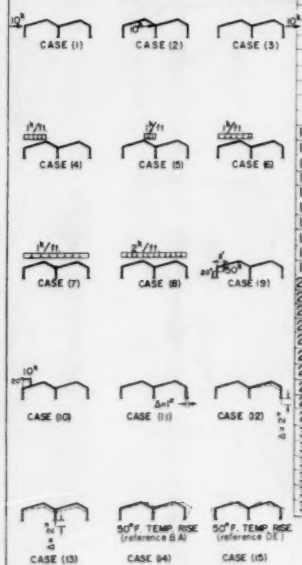


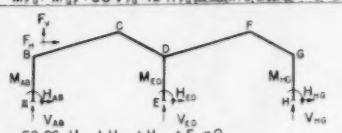
FIG. 6 Displacement of Foundations

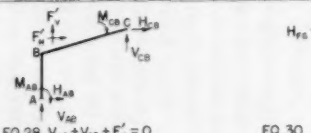
FIG. 7

LOADING CONDITIONS



NO.	UNSTRAINED EQUATIONS		LOAD CONSTANTS 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	$M_{AB} - 13.33\theta_A - 6.67\theta_B + 0.667\Delta_1$	=	0	0	0	0	0	0	0	0	+33.33	0	0	0	0	0	+4.20
1	$M_{BA} - 6.67\theta_A - 13.33\theta_B + 0.667\Delta_1$	=	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+4.20
2	$M_{ED} - 13.33\theta_E - 6.67\theta_D + 0.667\Delta_2$	=	0	0	0	0	0	0	0	0	0	0	0	0	0	-4.20	0
3	$M_{DE} - 6.67\theta_E - 13.33\theta_D + 0.667\Delta_2$	=	0	0	0	0	0	0	0	0	0	0	0	0	0	-4.20	0
4	$M_{HG} - 13.33\theta_H - 6.67\theta_G + 0.667\Delta_3$	=	0	0	0	0	0	0	0	0	0	0	+11.574	0	0	-8.40	-4.20
5	$M_{GH} - 6.67\theta_H - 13.33\theta_G + 0.667\Delta_3$	=	0	0	0	0	0	0	0	0	0	0	+11.574	0	0	-8.40	-4.20
6	$M_{BC} - 19.40\theta_B - 9.70\theta_C - 0.647\Delta_1 + 0.647\Delta_2$	=	0	0	0	-300	0	-300	-300	-600	0	-88.888	0	0	-11.25	0	0
7	$M_{CB} - 9.70\theta_B - 19.40\theta_C - 0.647\Delta_1 + 0.647\Delta_2$	=	0	0	0	+300	0	+300	+300	+600	0	+44.444	0	0	-11.25	0	0
8	$M_{CD} - 35.78\theta_C - 17.89\theta_D + 2.385\Delta_1 - 2.385\Delta_2$	=	0	0	0	0	-75	-75	-75	-150	0	0	0	0	-20.80	0	0
9	$M_{DC} - 17.89\theta_C - 35.78\theta_D + 2.385\Delta_1 - 2.385\Delta_2$	=	0	0	0	0	+75	+75	+75	+150	0	0	0	0	-20.80	0	0
10	$M_{DF} - 19.40\theta_D - 9.70\theta_F - 0.647\Delta_2 + 0.647\Delta_3$	=	0	0	0	0	0	0	-300	-600	0	0	0	-11.25	+11.25	0	0
11	$M_{FD} - 9.70\theta_D - 19.40\theta_F - 0.647\Delta_2 + 0.647\Delta_3$	=	0	0	0	0	0	0	+300	+600	0	0	0	-11.25	+11.25	0	0
12	$M_{FG} - 35.78\theta_F - 17.89\theta_G + 2.385\Delta_2 - 2.385\Delta_3$	=	0	0	0	0	0	0	-75	-150	0	0	0	-20.80	+20.80	0	0
13	$M_{GF} - 17.89\theta_F - 35.78\theta_G + 2.385\Delta_2 - 2.385\Delta_3$	=	0	0	0	0	0	0	+75	+150	0	0	0	-20.80	+20.80	0	0
14	$M_{BA} + M_{BC}$ $\Sigma M = 0$ AT JOINT B	=	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	$V_{CB} + V_{CD}$ $\Sigma V = 0$	=	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	$H_{CB} + H_{CD}$ $\Sigma H = 0$ AT JOINT C	=	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	$M_{CB} + M_{CD}$ $\Sigma M = 0$	=	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	$M_{DC} + M_{DE} + M_{DF}$ $\Sigma M = 0$ AT JOINT D	=	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	$V_{FD} + V_{FG}$ $\Sigma V = 0$	=	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	$H_{FD} + H_{FG}$ $\Sigma H = 0$ AT JOINT F	=	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	$M_{FD} + M_{FG}$ $\Sigma M = 0$	=	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	$M_{GF} + M_{GH}$ $\Sigma M = 0$ AT JOINT G	=	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	$M_{AB} + M_{BA} - 30H_{AB}$ COL AB, MOMENTS abt B	=	0	0	0	0	0	0	0	0	-100	0	0	0	0	0	0
24	$M_{ED} + M_{DE} - 30H_{ED}$ COL ED, MOMENTS abt D	=	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
25	$M_{HG} + M_{GH} - 30H_{HG}$ COL HG, MOMENTS abt G	=	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	$H_{AB} + H_{ED} + H_{HG}$ $\Sigma H = 0$ COMPLETE STRUCT.	=	-10	-10	-10	0	0	0	0	0	0	0	0	0	0	0	0
27	$V_{AB} + V_{ED} + V_{HG}$ $\Sigma V = 0$ COMPLETE STRUCT.	=	0	0	0	+60	+30	+90	+180	+360	+50	+10	0	0	0	0	0
28	$V_{AB} + V_{CB}$ $\Sigma V = 0$	=	0	0	0	60	0	60	60	120	+50	+10	0	0	0	0	0
29	$H_{AB} + H_{CB}$ $\Sigma H = 0$ SECTION ABC	=	-10	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	$V_{HG} + V_{FG}$ $\Sigma V = 0$	=	0	0	0	0	0	0	+30	+60	0	0	0	0	0	0	0
31	$H_{HG} + H_{FG}$ $\Sigma H = 0$ SECTION HGF	=	0	0	-10	0	0	0	0	0	0	0	0	0	0	0	0
32	$M_{BC} + M_{CB} - 60V_{CB} + 15H_{CB}$ (MEMBER BC, Mabt B)	=	0	0	0	-1800	0	-1800	-1800	-3600	0	-200	0	0	0	0	0
33	$M_{CD} + M_{DC} + 30V_{CD} + 15H_{CD}$ (MEMBER CD, Mabt D)	=	0	0	0	0	+450	+450	+450	+900	0	0	0	0	0	0	0
34	$M_{DF} + M_{FD} - 60V_{FD} + 15H_{FD}$ (MEMBER DF, Mabt D)	=	0	0	0	0	0	0	-1800	-3600	0	0	0	0	0	0	0
35	$M_{FG} + M_{GF} + 30V_{FG} + 15H_{FG}$ (MEMBER FG, Mabt G)	=	0	0	0	0	0	0	+450	+900	0	0	0	0	0	0	0





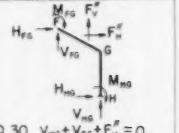
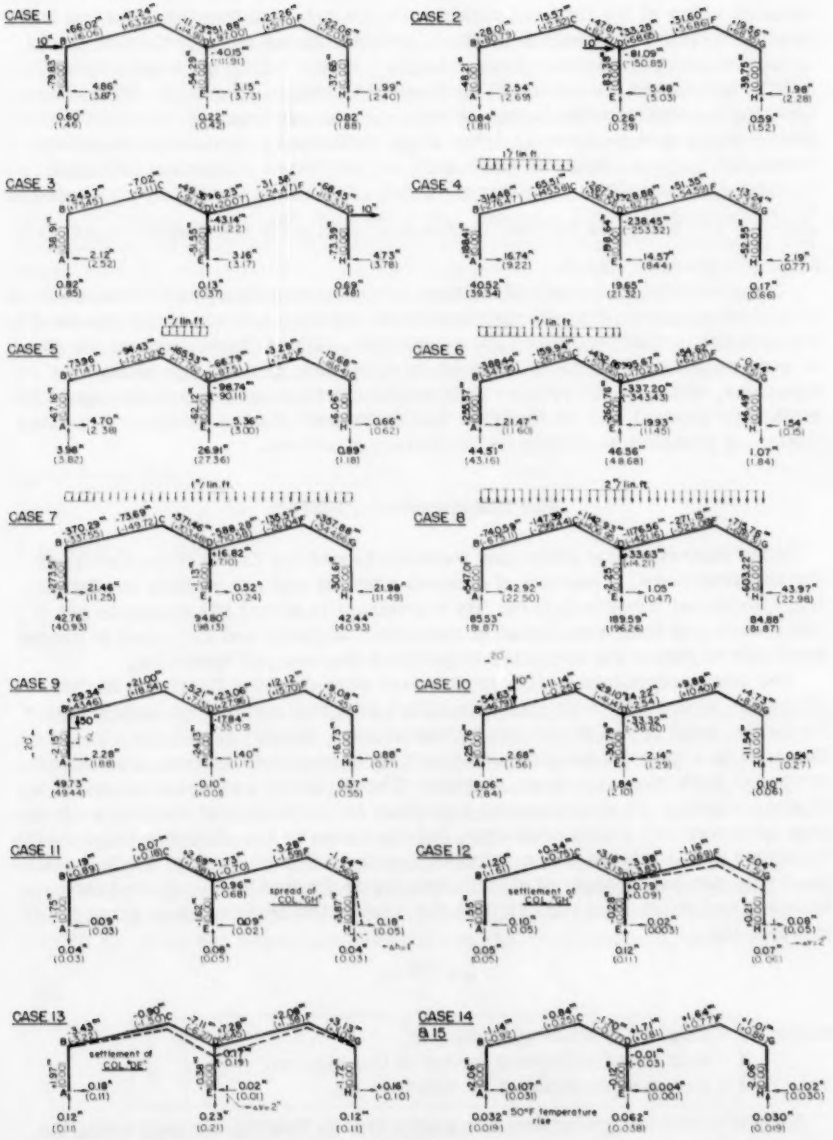


TABLE I - COLUMN BASES ASSUMED FIXED

	CASE 1	CASE 2	CASE 3	CASE 4	CASE 5	CASE 6	CASE 7	CASE 8	CASE 9	CASE 10	CASE 11	CASE 12	CASE 13	CASE 14	CASE 15
							END MOMENTS - (FT. KIPS)				(MULTIPLY BY $\frac{1}{100}$ FOR CASES 11 THROUGH 15)				
M_{AB}	- 79.83	- 48.21	- 38.91	+188.41	+ 67.16	+255.57	+273.51	+547.01	- 2.15	+ 25.76	- 1.75	- 1.55	+ 1.97	+ 2.06	+ 2.06
M_{BA}	- 54.29	- 83.38	- 51.55	-198.64	- 62.12	-260.76	- 1.13	- 2.25	- 24.13	- 30.79	- 1.62	- 0.28	- 0.38	- 0.12	- 0.12
M_{BC}	- 37.65	- 39.75	- 73.39	- 52.55	+ 6.06	- 46.49	-301.61	-603.22	- 16.73	- 11.54	+ 3.87	+ 0.21	- 1.72	- 2.06	- 2.06
M_{CB}	- 66.02	- 28.01	- 24.57	+314.48	+ 73.96	+388.44	+370.29	+740.59	- 29.34	+ 54.63	- 1.19	- 1.20	+ 3.43	+ 1.14	+ 1.14
M_{BE}	+ 66.02	+ 28.01	+ 24.57	-314.48	- 73.96	-388.44	-370.29	-740.59	+ 29.34	- 54.63	+ 1.19	+ 1.20	- 3.43	- 1.14	- 1.14
M_{CD}	+ 47.24	- 15.57	- 7.02	- 65.51	- 94.43	-159.94	- 73.69	-147.39	+ 21.00	+ 11.14	- 0.07	+ 0.34	- 0.90	+ 0.84	+ 0.84
M_{CG}	- 47.24	+ 15.57	+ 7.02	+ 65.51	+ 94.43	+159.94	+ 73.69	+147.39	- 21.00	- 11.14	+ 0.07	- 0.34	+ 0.90	- 0.84	- 0.84
M_{DC}	- 11.73	+ 47.81	+ 49.36	+267.33	+165.53	+432.86	+571.46	+1142.93	- 5.21	+ 29.10	+ 2.69	+ 3.18	- 7.11	- 1.70	- 1.70
M_{DE}	+ 51.88	+ 33.28	- 6.23	- 28.88	- 66.79	- 95.67	-588.28	-1176.56	+ 23.06	+ 4.22	- 1.73	- 3.98	+ 7.28	+ 1.71	+ 1.71
M_{EB}	- 40.15	- 81.09	- 43.14	-238.45	- 98.74	-337.20	+ 16.82	+ 33.63	- 17.84	- 33.32	- 0.96	+ 0.79	- 0.17	- 0.01	- 0.01
M_{ED}	+ 27.26	+ 31.60	- 31.38	+ 51.35	+ 3.28	+ 54.62	-135.57	-271.15	+ 12.12	+ 9.88	- 3.28	- 1.16	+ 2.08	+ 1.64	+ 1.64
M_{FG}	- 27.26	- 31.60	+ 31.38	- 51.35	- 3.28	- 54.62	+135.57	+271.15	- 12.12	- 9.88	+ 3.28	+ 1.16	- 2.08	- 1.64	- 1.64
M_{GF}	+ 22.06	+ 19.56	+ 68.45	+ 13.24	- 13.68	- 0.44	+357.88	+715.76	+ 9.80	+ 4.73	- 1.64	- 2.04	+ 3.13	+ 1.01	+ 1.01
M_{GH}	- 22.06	- 19.56	- 68.45	- 13.24	+ 13.68	+ 0.44	-357.88	-715.76	- 9.80	- 4.73	+ 1.64	+ 2.04	- 3.13	- 1.01	- 1.01
							VERTICAL SHEARS - (KIPS)				(MULTIPLY BY $\frac{1}{100}$ FOR CASES 11 THROUGH 15)				
V_{AB}	- 0.60	- 0.84	- 0.82	+ 40.52	+ 3.98	+ 44.51	+ 42.76	+ 85.53	+ 49.73	+ 8.06	- 0.04	- 0.05	+ 0.12	+ 0.03	+ 0.03
V_{EB}	- 0.22	+ 0.26	+ 0.13	+ 19.65	+ 26.91	+ 46.56	+ 94.80	+189.59	- 0.10	+ 1.84	+ 0.08	+ 0.12	- 0.23	- 0.06	- 0.06
V_{BC}	+ 0.82	+ 0.59	+ 0.69	- 0.17	- 0.89	- 1.07	+ 42.44	+ 84.88	+ 0.37	+ 0.10	- 0.04	- 0.07	+ 0.12	+ 0.03	+ 0.03
V_{CB}	+ 0.60	+ 0.84	+ 0.82	+ 19.48	- 3.98	+ 15.49	+ 17.24	+ 34.47	+ 0.27	+ 1.94	+ 0.04	+ 0.05	- 0.12	- 0.03	- 0.03
V_{CD}	- 0.60	- 0.84	- 0.82	- 19.48	+ 3.98	- 15.49	- 17.24	- 34.47	- 0.27	- 1.94	- 0.04	- 0.05	+ 0.12	+ 0.03	+ 0.03
V_{FD}	+ 0.82	+ 0.59	+ 0.69	- 0.17	- 0.89	- 1.07	+ 12.44	+ 24.88	+ 0.37	+ 0.10	- 0.04	- 0.07	+ 0.12	+ 0.03	+ 0.03
V_{FG}	- 0.82	- 0.59	- 0.69	+ 0.17	+ 0.89	+ 1.07	- 12.44	- 24.88	- 0.37	- 0.10	+ 0.04	+ 0.07	- 0.12	- 0.03	- 0.03
							HORIZONTAL SHEARS - (KIPS)				(MULTIPLY BY $\frac{1}{100}$ FOR CASES 11 THROUGH 15)				
H_{AB}	- 4.86	- 2.54	- 2.12	+ 16.76	+ 4.70	+ 21.47	+ 21.46	+ 42.92	+ 2.28	+ 2.68	- 0.09	- 0.10	+ 0.18	+ 0.11	+ 0.11
H_{ED}	- 3.15	- 5.48	- 3.16	- 14.57	- 5.36	- 19.93	+ 0.52	+ 1.05	- 1.40	- 2.14	+ 0.09	+ 0.02	- 0.02	- 0.004	- 0.004
H_{BC}	- 1.99	- 1.98	- 4.73	- 2.19	+ 0.66	- 1.54	- 21.98	- 43.97	- 0.88	- 0.54	+ 0.18	+ 0.08	- 0.16	- 0.10	- 0.10
H_{CB}	- 5.14	+ 2.54	+ 2.12	- 16.76	- 4.70	- 21.47	- 21.46	- 42.92	- 2.28	- 2.68	+ 0.10	+ 0.09	- 0.18	- 0.11	- 0.11
H_{CD}	+ 5.14	- 2.54	- 2.12	+ 16.76	+ 4.70	+ 21.47	+ 21.46	+ 42.92	+ 2.28	+ 2.68	- 0.10	- 0.09	+ 0.18	+ 0.11	+ 0.11
H_{FD}	- 1.99	- 1.98	- 5.27	- 2.19	+ 0.66	- 1.54	- 21.98	- 43.97	- 0.88	- 0.54	+ 0.18	+ 0.08	- 0.16	- 0.10	- 0.10
H_{FG}	+ 1.99	+ 1.98	- 5.27	+ 2.19	- 0.66	+ 1.54	+ 21.98	+ 43.97	+ 0.88	+ 0.54	- 0.18	- 0.08	+ 0.16	+ 0.10	+ 0.10
							JOINT ROTATIONS - ($\frac{0.48}{I}$ RADIANS)								
θ_A	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
θ_E	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
θ_H	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
θ_B	+ 2.074	+ 3.034	+ 2.153	+ 18.929	+ 1.021	+ 19.951	+ 14.533	+ 29.066	+ 0.922	+ 4.335	+ 0.084	+ 0.053	+ 0.219	- 0.139	- 0.139
θ_C	+ 0.138	- 1.458	- 1.104	- 17.259	- 1.089	- 18.348	- 16.745	- 33.491	+ 0.061	- 2.630	- 0.046	- 0.036	+ 0.480	+ 0.065	+ 0.065
θ_D	+ 2.123	+ 0.344	+ 1.263	- 5.977	- 5.499	- 11.476	- 2.694	- 5.388	+ 0.943	+ 0.380	+ 0.100	+ 0.161	+ 0.032	+ 0.017	+ 0.017
θ_F	- 0.415	+ 0.171	- 1.330	+ 2.293	- 1.724	+ 4.017	- 12.491	- 24.982	- 0.185	+ 0.204	- 0.060	+ 0.452	- 0.504	+ 0.009	+ 0.009
θ_G	+ 2.341	+ 3.031	+ 0.742	+ 5.903	+ 1.143	+ 7.046	- 8.449	- 16.899	+ 1.041	+ 1.021	- 0.334	+ 0.274	- 0.212	+ 0.157	+ 0.157
							COLUMN DEFLECTIONS - ($\frac{0.48}{I}$ FEET)								
Δ_1	+140.42	+102.63	+ 79.87	- 93.18	- 90.48	-183.66	-264.73	-529.45	+ 62.41	+ 4.74	+ 3.45	+ 2.86	- 0.77	- 4.49	+ 1.81
Δ_2	+102.62	+128.45	+ 89.91	+238.04	+ 38.14	+276.19	+ 28.63	+ 57.25	+ 45.61	+ 42.37	+ 3.43	+ 2.04	+ 0.90	- 5.94	+ 0.35
Δ_3	+ 79.86	+ 89.90	+117.45	+137.83	+ 2.33	+140.16	+367.69	+735.39	+ 35.49	+ 27.50	+ 8.21	+ 2.41	+ 0.49	- 7.94	- 1.65

TABLE II – COLUMN BASES ASSUMED HINGED

	CASE 1	CASE 2	CASE 3	CASE 4	CASE 5	CASE 6	CASE 7	CASE 8	CASE 9	CASE 10	CASE 11	CASE 12	CASE 13	CASE 14	CASE 15
							END MOMENTS – (FT. KIPS)			(MULTIPLY BY $\frac{1}{100}$ FOR CASES 11 THROUGH 15)					
M_{AB}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M_{EC}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M_{HG}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M_{BA}	-116.06	- 80.79	- 75.45	+276.47	+ 71.47	+347.95	+337.55	+ 675.11	- 43.46	+ 46.79	- 0.89	- 1.61	+ 3.22	+ 0.92	+ 0.92
M_{BC}	+116.06	+ 80.79	+ 75.45	- 276.47	- 71.47	-347.95	-337.55	- 675.11	+ 43.46	- 46.79	+ 0.89	+ 1.61	- 3.22	- 0.92	- 0.92
M_{CB}	+ 63.22	- 12.52	- 2.11	- 145.58	-122.02	-267.60	-149.72	- 299.44	+ 18.54	- 0.25	+ 0.18	+ 0.75	- 1.50	+ 0.25	+ 0.25
M_{CD}	- 63.22	+ 12.52	+ 2.11	+ 145.58	+122.02	+267.60	+149.72	+ 299.44	- 18.54	+ 0.25	- 0.18	- 0.75	+ 1.50	- 0.25	- 0.25
M_{DC}	+ 14.91	+ 82.21	+ 91.14	+336.04	+177.62	+513.66	+703.48	+1406.95	+ 7.13	+ 41.14	+ 1.38	+ 3.13	- 6.27	- 0.77	- 0.77
M_{DF}	+ 97.00	+ 68.65	+ 20.07	- 82.72	- 87.51	- 170.23	- 710.58	-1421.16	+ 27.96	- 2.54	- 0.70	- 3.23	+ 6.45	+ 0.81	+ 0.81
M_{DE}	-111.91	-150.85	-111.22	-253.32	- 90.11	- 343.43	+ 7.10	+ 14.21	- 35.09	- 38.60	+ 0.68	+ 0.09	- 0.19	- 0.03	- 0.03
M_{FD}	+ 51.70	+ 56.86	- 24.47	+ 54.59	+ 7.42	+ 62.01	- 261.04	- 522.08	+ 15.70	+ 10.40	- 1.59	- 0.69	+ 1.38	+ 0.77	+ 0.77
M_{FG}	- 51.70	- 56.86	+ 24.47	- 54.59	- 7.42	- 62.01	+ 261.04	+ 522.08	- 15.70	- 10.40	+ 1.59	+ 0.69	- 1.38	- 0.77	- 0.77
M_{GF}	+ 72.03	+ 68.35	+113.33	+ 23.16	+ 18.64	+ 4.51	+ 344.66	+ 689.31	+ 21.45	+ 8.19	+ 1.56	+ 1.51	+ 3.03	+ 0.88	+ 0.89
M_{GH}	- 72.03	- 68.35	-113.33	- 23.16	+ 18.64	- 4.51	- 344.66	- 689.31	- 21.45	- 8.19	+ 1.56	+ 1.51	- 3.03	- 0.88	- 0.89
							VERTICAL SHEARS – (KIPS)			(MULTIPLY BY $\frac{1}{100}$ FOR CASES 11 THROUGH 15)					
V_{AB}	- 1.46	- 1.81	- 1.85	+ 39.34	+ 3.82	+ 43.16	+ 40.93	+ 81.87	+ 49.44	+ 7.84	- 0.03	- 0.05	+ 0.11	+ 0.02	+ 0.02
V_{ED}	- 0.42	+ 0.29	+ 0.37	+ 21.32	+ 27.36	+ 48.68	+ 98.13	+ 196.26	+ 0.01	+ 2.10	+ 0.05	+ 0.11	- 0.21	- 0.04	- 0.04
V_{HG}	+ 1.88	+ 1.52	+ 1.48	- 0.66	- 1.18	- 1.84	+ 40.93	+ 81.87	+ 0.55	+ 0.06	- 0.03	- 0.06	+ 0.11	+ 0.02	+ 0.02
V_{CB}	+ 1.46	+ 1.81	+ 1.85	+ 20.66	- 3.82	+ 16.84	+ 19.07	+ 38.13	+ 0.56	+ 2.16	+ 0.03	+ 0.05	- 0.11	- 0.02	- 0.02
V_{CD}	- 1.46	- 1.81	- 1.85	- 20.66	+ 3.82	- 16.84	- 19.07	- 38.13	- 0.56	- 2.16	- 0.03	- 0.05	+ 0.11	+ 0.02	+ 0.02
V_{FD}	+ 1.88	+ 1.52	+ 1.48	- 0.66	- 1.18	- 1.84	+ 10.93	+ 21.87	+ 0.55	+ 0.06	- 0.03	- 0.05	+ 0.11	+ 0.02	+ 0.02
V_{FG}	- 1.88	- 1.52	- 1.48	+ 0.66	+ 1.18	+ 1.84	- 10.93	- 21.87	- 0.55	- 0.06	+ 0.03	+ 0.05	- 0.11	- 0.02	- 0.02
							HORIZONTAL SHEARS – (KIPS)			(MULTIPLY BY $\frac{1}{100}$ FOR CASES 11 THROUGH 15)					
H_{AB}	- 3.87	- 2.69	- 2.52	+ 9.22	+ 2.38	+ 11.60	+ 11.25	+ 22.50	+ 1.88	+ 1.56	- 0.03	- 0.05	+ 0.11	+ 0.03	+ 0.03
H_{ED}	- 3.73	- 5.03	- 3.71	- 8.44	- 3.00	- 11.45	+ 0.24	+ 0.47	- 1.17	- 1.29	+ 0.02	+ 0.003	- 0.016	- 0.001	- 0.001
H_{HG}	- 2.40	- 2.28	- 3.78	- 0.77	+ 0.62	- 0.15	- 11.49	- 22.98	- 0.71	- 0.27	+ 0.05	+ 0.05	- 0.10	- 0.03	- 0.03
H_{CB}	- 6.13	+ 2.69	+ 2.52	- 9.22	- 2.38	- 11.60	- 11.25	- 22.50	- 1.88	- 1.56	+ 0.03	+ 0.05	- 0.11	- 0.03	- 0.03
H_{CD}	+ 6.13	- 2.69	- 2.52	+ 9.22	+ 2.38	+ 11.60	+ 11.25	+ 22.50	+ 1.88	+ 1.56	- 0.03	- 0.05	+ 0.11	+ 0.03	+ 0.03
H_{FD}	- 2.40	+ 2.28	+ 6.22	- 0.77	- 0.62	- 0.15	- 11.49	- 22.98	+ 0.71	- 0.27	+ 0.05	+ 0.05	- 0.10	- 0.03	- 0.03
H_{FG}	+ 2.40	+ 2.28	- 6.22	+ 0.77	- 0.62	+ 0.15	+ 11.49	+ 22.98	- 0.71	+ 0.27	- 0.05	- 0.05	+ 0.10	+ 0.03	+ 0.03
							JOINT ROTATIONS – ($\frac{.048}{I}$ RADIANS)								
θ_A	+ 22.234	+ 18.972	+ 16.960	- 14.381	- 7.662	- 22.043	- 29.046	- 58.091	+ 3.669	- 1.341	+ 0.183	+ 0.298	- 0.210	- 0.225	- 0.225
θ_E	+ 20.530	+ 23.767	+ 19.311	+ 27.453	+ 6.112	+ 33.565	+ 0.648	+ 1.297	+ 6.283	+ 4.817	+ 0.166	+ 0.172	+ 0.043	+ 0.010	+ 0.010
θ_H	+ 16.788	+ 17.164	+ 20.003	+ 11.396	- 1.058	+ 10.338	+ 34.244	+ 68.487	+ 5.048	+ 2.622	- 0.438	+ 0.097	+ 0.194	+ 0.228	+ 0.228
θ_B	+ 4.807	+ 6.841	+ 5.631	+ 27.132	+ 3.069	+ 30.201	+ 21.638	+ 43.276	+ 2.148	+ 5.684	+ 0.050	+ 0.057	+ 0.273	+ 0.087	+ 0.087
θ_C	- 0.641	- 2.779	- 2.365	- 21.229	- 2.142	- 23.371	- 20.854	- 41.707	- 0.421	- 3.264	- 0.023	- 0.031	+ 0.449	+ 0.034	+ 0.034
θ_D	+ 3.727	+ 1.116	+ 2.611	- 10.583	- 7.419	- 18.002	- 1.715	- 3.430	- 1.014	- 0.979	+ 0.064	+ 0.186	+ 0.015	+ 0.005	+ 0.005
θ_F	- 0.943	- 0.099	- 1.981	+ 3.573	+ 2.368	+ 5.941	- 13.796	- 27.592	+ 0.249	+ 0.354	- 0.027	+ 0.447	- 0.507	+ 0.002	+ 0.002
θ_G	+ 5.973	+ 6.900	+ 2.986	- 7.919	+ 1.74	- 9.650	- 17.507	- 35.013	+ 1.827	+ 1.393	- 0.204	+ 0.324	- 0.261	+ 0.094	+ 0.094
							COLUMN DEFLECTIONS – ($\frac{.048}{I}$ FEET)								
Δ_1	+492.41	+447.56	+395.26	- 16.08	-122.43	- 138.51	- 364.10	- 728.10	+144.78	+ 30.04	+ 4.17	+ 6.53	- 1.48	- 5.37	+ 0.92
Δ_2	+447.56	+486.14	+412.04	+ 44.82	+ 47.98	+ 490.78	+ 30.11	+ 60.22	+135.71	+ 86.47	+ 3.95	+ 5.29	+ 1.01	- 6.04	+ 0.26
Δ_3	+395.23	+412.02	+429.62	+306.93	- 3.74	+303.19	+509.29	+1018.59	+119.16	+ 66.34	+ 6.56	+ 5.17	+ 1.28	- 7.09	- 0.79



NOTE: Multiply the numerical values of moments and shears by $\left(\frac{I_c}{100}\right)$ for cases 11 through 15, I_c being the moment of inertia of the columns in (inches)^4

FIG. 8

absolute rotations and deflections are needed. For a steel frame E = 30 x 10^6 psi and I is usually given in (inches)^4. Hence, the flexural rigidity in Kips and Feet, will be $E \times I = 30{,}000{,}000 \times \frac{144}{1000} \times \frac{I}{20736} = 208.333I$. The relative value of the flexural rigidity for the columns was taken as one hundred. To obtain rotations in radians and deflections in feet, the numerical values of the deformations in the results (Tables I & II) must be divided by 2.083I, where I is the columns' moment of inertia in (inches)^4. When considering foundations displacement and temperature changes, the load constants (fixed-end moments) for the slope deflection equations were determined for $I_{COL} = 100(in)^4$. Obviously, for any other numerical value of moment of inertia, the resulting moments and shears should be multiplied by $\frac{I_{COL}}{100}$. The formulas for the fixed-end moments for the displacement conditions are given in Fig. 6.

The possibilities for simplification have, by no means, been exhausted. It should be apparent, that geometric considerations need not always be used to express the displacements of the peak joints. The deflection terms for the sloping members may be introduced as unknowns, in the slope-deflection equations, which would require four additional equations of equilibrium. The method is general and its inherent flexibility permits the designer a limited degree of freedom in setting up the needed equations.

The Mathematical Program

Many mathematical programs were developed for the various electronic digital computers. They are of general interest and are readily available from computer manufacturers. By a program is meant the complete set of commands and data, expressed in computer language and arranged in proper sequence to direct the computer to perform the desired operation.

The linear equations in this paper were solved on the Datatron, digital computer model 205, with a mathematical program written by Jack Warga,* formerly, head of the Mathematics Department, Electrodata Corp. The Datatron is a general purpose, internally programmed, decimal, electronic computer with magnetic drum storage. The program was set up to solve, up to and including, 54 simultaneous equations for a variety of constants. In the case of structural frame problems, the constants in the right hand side of the equations are functions of the imposed loading. The number of loading conditions that can be considered simultaneously is limited by the size of the memory and decreases rapidly with the size of the system. It is given by equation (5),

$$m = \frac{3300}{a} - n \qquad (5)$$

where, m = number of loading conditions
n = number of unknowns (order of the system)
a = smallest multiple of 20 which is $\geq$ n

All mathematical operations are performed in floating decimal using the automatic floating point control unit. Floating point operation is a method of

*on a research project in Israel.

performing arithmetic computation with numbers which vary in magnitude over a relatively wide range. When using a slide rule or a desk calculator the designer himself has to keep track of the decimal point. Essentially, with the floating point operation, the position of the decimal point is translated to the machine language.

A number X may be represented by a pair of numbers Y and Z in the form $X = Y\ 10^Z$ where $0.1 \leq Y < 1.0$ and Z is an integer. The decimal fraction Y is called the mantissa, and the interger Z is called the exponent or power. For example, the number 16.124 can be written as $.16124 \times 10^2$. Here .16124 is the mantissa and the exponent to the base 10 is 2. Similarly the number .0016124 can be written as $.16124 \times 10^{-2}$. As can be seen from the examples, the exponent to the base 10 can be either positive or negative in sign. To eliminate having to express the sign of the exponent, fifty is added to the appropriate power of ten. The exponent may range from 00 to 99 which is interpreted as -50 and +49. Thus, any number which falls within the range 10^{-51} to 10^{+49} may be represented in the floating point structure. Zero is represented as zero exponent and zero mantissa. To further simplify presentation, the exponent (in floating point) precedes the mantissa. They need not be separated. Several examples of converting fixed point numbers to floating point representation are given below.

Fixed Point Number		Power of Ten	Floating Point Exponent	Floating Point Number
16.124	$= .16124 \times 10^2$	+2	52	5216124
0.0016124	$= .16124 \times 10^{-2}$	-2	48	4816124
161240	$= .16124 \times 10^6$	+6	56	5616124
0.16124	$= .16124 \times 10^0$	+0	50	5016124
1.6124	$= .16124 \times 10^1$	+1	51	5116124

The coefficients of the unknowns and the constants are arranged in matrix form. The unstrained equations constitute matrix A, and the constants are entered in matrix B. The matrices are arranged in consecutive columns, each column having assigned to it "a" locations (20, 40 or 60). The main memory of the magnetic drum in the Datatron has a capacity of 4000 words. The length of each being 10 decimal digits and algebraic sign. (Fig. 9). It is convenient to use the first 700 locations (0000 to 0699) for the simultaneous equations program, and to start the first column of matrix A in location 0700. The first column of matrix B starts immediately after the last column of matrix A. Once the words are recorded on the drum surface, they remain

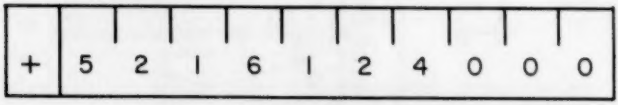


FIG. 9

TABLE III

TWO BAY GABLED RIGID FRAME

Column Bases Assumed FIXED

(NOTE For bases HINGED, retain θ_A, θ_B, θ_H, and cross out M_{AB}, M_{BA}, M_{HG})

MATRIX DICTIONARY

4 0000	PTR	6000	
0 0000	00	0035	(M−1 Equations)
0 0000	00	0014	(M−1 Conditions)
4 0000	PTR	6017	
0 0000	00	0700	
0 0000	00	2140	(N Columns of Conditions)
0 0000	00	2140	(N Columns of Results)
8 0000	CUR	0000	

MATRIX A

MATRIX B

LOADING CONDITIONS

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

there (whether or not power is turned on) until other words are written over them. Memory locations 0700-3999 are, therefore, zeroed prior to read-in of the data to "erase" any old information that might have been left over from a prior problem, and to reduce input data to non-zero elements only. It is unnecessary to preserve matrix B, and the solutions may be stored in the same memory domain of B, for either use in a subsequent problem, or transmitted to an output media, which may be a Flexowriter (electric typewriter), punched paper tape, punched cards or magnetic tape. The program also provides for the measure of error, by substituting the solution in an auxiliary equation which is the sum of all the equations of the system. The difference between the thus calculated left-hand side and the right-hand side is divided by the right hand side, and printed out as an indication of error. The error recorded for the frame of example 1 was 0.003 percent. Programs for solving larger systems are available for large scale computers. The IBM 704 may be used for simultaneous equations with hundreds of unknowns. However, precision is inversely proportional to the order of the system, and special control features must be incorporated to assure a reasonable degree of accuracy in the end results, due to round-of-errors.

Frames with Tapered Girders

For members with variable moment of inertia, the slope-deflection equations are expressed in the general form by equations 6 and 7 (See Fig. 10).

$$M_{AB} = \frac{EI_0}{L}\left[k_{AA}\theta_A + k_{BA}\theta_B - \left(k_{AA}+k_{BA}\right)\frac{\Delta}{L}\right] + M'_{AB} \tag{6}$$

$$M_{BA} = \frac{EI_0}{L}\left[k_{AB}\theta_A + k_{BB}\theta_B - \left(k_{BB}+k_{AB}\right)\frac{\Delta}{L}\right] + M'_{BA} \tag{7}$$

where

E = modulus of elasticity

I_0 = moment of inertia of section of minimum depth

L = length of member

k_{AA} = stiffness factor of member AB at end A; stiffness = $k_{AA}\frac{EI_0}{L}$

k_{BB} = stiffness factor of member AB at end B; $K_{BB} = k_{BB}\frac{EI_0}{L}$

$k_{AB} = k_{AA} \times C_{AB}$
$k_{BA} = k_{BB} \times C_{BA}$
} $k_{AB} = k_{BA}$ (from Maxwell's Law of Reciprocity)

C_{AB} = carry-over factor of member AB at end A

C_{BA} = carry-over factor of member AB at end B

M_{AB} = restraining moment at end A

M_{BA} = restraining moment at end B

M'_{AB} = fixed end moment at end A, due to loading of member AB

M'_{BA} = fixed end moment at end B, due to loading of member AB

θ_A = rotation at end A

θ_B = rotation at end B

Δ = $\delta_B - \delta_A$, relative displacement of end supports

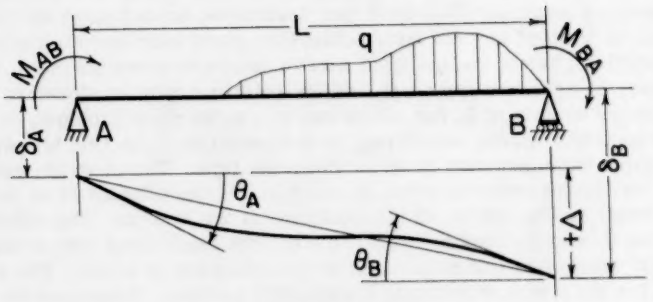


FIG. 10

Again, it is convenient to group the unknowns to the left-hand side, and the equations may be written:

$$M_{AB} - C_1\theta_A - C_2\theta_B + C_4\Delta = + M'_{AB} \qquad (8)$$

$$M_{BA} - C_2\theta_A - C_3\theta_B + C_5\Delta = + M'_{BA} \qquad (9)$$

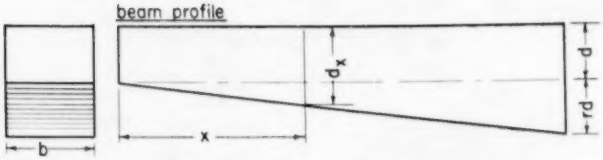
where

$$C_1 = \frac{EI_o}{L} k_{AA} \qquad C_2 = \frac{EI_o}{L} k_{AB} \qquad C_3 = \frac{EI_o}{L} k_{BB} \qquad C_4 = \frac{C_1 + C_2}{L} \qquad C_5 = \frac{C_3 + C_2}{L}$$

The calculation of the necessary constants for members with variable moment of inertia is rather a long and tedious matter. Frame constants for a variety of beam profiles and types of loading commonly met in practice are usually available from tables and charts in various technical publications. For tapered members, the variation of the moment of inertia is expressed as a simple mathematical equation that is easily integrable. The expressions for the constants are given in Fig. 11. The upper case letters of the subscript are used to designate the joint, while the lower case relates to the type of loading. When "r" is less than unity, the Maclaurin's series expansion of the logarithmic function and algebraic simplifications yield formulas that are considerably easier to evaluate. Only the first few terms of the series are significant. For the limiting value, as "r" approaches zero, the beam is straight, and the expressions yield results for the simple case of the constant moment of inertia. Numerical values for frame constants for tapered members were computed on the Datatron. After the necessary coefficients are evaluated, the procedure involved in determining the unknown quantities for frames containing members with variable moment of inertia, is exactly the same as that used in constant I frameworks. The procedure will now be applied to a frame containing tapered roof girders, under various loading conditions.

The frame of example 1 was chosen to illustrate the method, rather than to present detailed calculations related to an actual structure, and the relative rigidities rather than absolute properties, were conveniently used. The

End Rotations of Tapered Beams (flexure only)



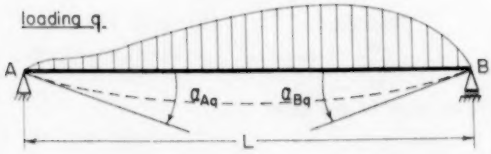
$$d_x = d\left(1 + r\frac{X}{L}\right) = d(1 + rz)$$

$$z = \frac{X}{L}$$

$$I_o = \frac{1}{12} bd^3$$

$$I_x = I_o(1 + rz)^3$$

E = constant

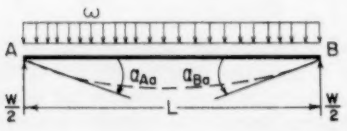


any loading q on simply supported beam AB

angular rotation of end tangent at A, $\alpha_{Aq} = \frac{1}{L}\int_B^A \frac{M_x(L-x)dx}{EI_x}$

angular rotation of end tangent at B, $\alpha_{Bq} = \frac{1}{L}\int_B^A \frac{M_x x\,dx}{EI_x}$

Loading "a". uniformly distributed load



$$W = \omega L$$

$$M_X = \frac{WL}{2}\left(\frac{X}{L}\right)\left(1 - \frac{X}{L}\right)$$

$$c_{Aa} = \frac{6+r}{4r^3} - \frac{3+2r}{2r^4}\,\mathrm{Log}_e(1+r)$$

$$c_{Ba} = \frac{6+5r}{4r^3(1+r)} - \frac{3+r}{2r^4}\,\mathrm{Log}_e(1+r)$$

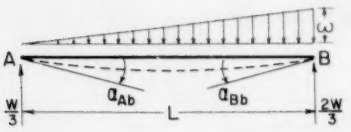
$$\alpha_{Aa} = \frac{WL^2}{EI_o} c_{Aa}$$

$$\alpha_{Ba} = \frac{WL^2}{EI_o} c_{Ba}$$

$r \le 1.0$	$r = 0$
$c_{Aa} = \sum_{n=1} \frac{(-1)^{n+1} r^{n-1}}{2(n+2)(n+3)}$	$C_{Aa} = +\frac{1}{24}$
$c_{Ba} = -\frac{1}{4(1+r)} + \sum_{n=1} \frac{(-1)^{n+1}(3+2n)r^{n-1}}{2(n+2)(n+3)}$	$C_{Ba} = -\frac{1}{24}$

FIG. 11

Loading "b" Load increasing uniformly to deeper end



$$W = \frac{1}{2}\omega L$$

$$M_x = \frac{WL}{3}\left(\frac{X}{L}\right)\left[1-\left(\frac{X}{L}\right)^2\right]$$

$$C_{Ab} = \frac{r^2-12}{6r^4} + \frac{(6+3r-r^2)}{3r^5}\,\mathrm{Log_e}(1+r)$$

$$C_{Bb} = \frac{(2r^2-3r-6)}{3r^4(1+r)} + \frac{(6-r^2)}{3r^5}\,\mathrm{Log_e}(1+r)$$

$$\alpha_{Ab} = \frac{WL^2}{EI_o}C_{Ab}$$

$$\alpha_{Bb} = \frac{WL^2}{EI_o}C_{Bb}$$

$r \le 1.0$

$$C_{Ab} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n(2n+5)r^{n-1}}{3(n+2)(n+3)(n+4)}$$

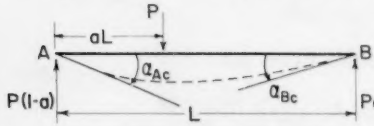
$$C_{Bb} = -\frac{1}{3(1+r)} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(5n+8)r^{n-1}}{3(n+2)(n+4)}$$

$r = 0$

$$C_{Ab} = +\frac{7}{180}$$

$$C_{Bb} = -\frac{2}{45}$$

Loading "c" Concentrated load at any point



$$M_x = \begin{cases} PL(1-a)\frac{X}{L} & X<aL \\ PLa(1-\frac{X}{L}) & X>aL \end{cases}$$

$$C_{Ac} = \frac{a(1-a)}{2r(1+ra)} + \frac{1}{r^3}\,\mathrm{Log_e}\frac{(1+r)^a}{(1+ra)}$$

$$C_{Bc} = \frac{a(1-a)}{2r(1+r)(1+ra)} + \frac{1}{r^3}\,\mathrm{Log_e}\frac{(1+r)^a}{(1+ra)}$$

$$\alpha_{Ac} = \frac{PL^2}{EI_o}C_{Ac}$$

$$\alpha_{Bc} = \frac{PL^2}{EI_o}C_{Bc}$$

$r \le 1.0$

$$C_{Ac} = -\frac{a^2(1-a)}{2(1+ra)} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}ar^{n-1}(1-a^{n+1})}{n+2}$$

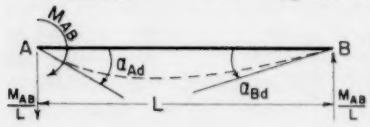
$$C_{Bc} = +\frac{a^3}{2(1+ra)} - \frac{a}{2(1+r)} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}ar^{n-1}(1-a^{n+1})}{n+2}$$

$r = 0$

$$C_{Ac} = +\frac{a(1-a)(2-a)}{6}$$

$$C_{Bc} = -\frac{a(1-a^2)}{6}$$

Loading "d" Moment M_{AB} applied at end A



$$M_x = M_{AB}\left(1-\frac{X}{L}\right)$$

$$C_{Ad} = -\frac{2-r}{2r^2} + \frac{1}{r^3}\,\mathrm{Log_e}(1+r)$$

$$C_{Bd} = -\frac{2+r}{2r^2(1+r)} + \frac{1}{r^3}\,\mathrm{Log_e}(1+r)$$

$$\alpha_{Ad} = \frac{M_{AB}L}{EI_o}C_{Ad}$$

$$\alpha_{Bd} = \frac{M_{AB}L}{EI_o}C_{Bd}$$

$r \le 1.0$

$$C_{Ad} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}r^{n-1}}{n+2}$$

$$C_{Bd} = -\frac{1}{2(1+r)} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}r^{n-1}}{n+2}$$

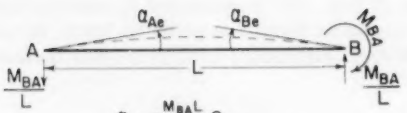
$r = 0$

$$C_{Ad} = +\frac{1}{3}$$

$$C_{Bd} = -\frac{1}{6}$$

FIG. 11 (Cont.)

Loading "e" Moment M_{BA} applied at end B



$$M_x = -M_{BA}\frac{X}{L}$$

$$C_{Ae} = -\frac{2+r}{2r^2(1+r)} + \frac{1}{r^3}\text{Log}_e(1+r)$$

$$C_{Be} = -\frac{2+3r}{2r^2(1+r)^2} + \frac{1}{r^3}\text{Log}_e(1+r)$$

$$\alpha_{Ae} = \frac{M_{BA}L}{EI_o}C_{Ae}$$

$$\alpha_{Be} = \frac{M_{BA}L}{EI_o}C_{Be}$$

$r \le 1.0$ $\qquad$ $r = 0$

Note. $C_{Bd} = C_{Ae}$

$$C_{Ae} = -\frac{1}{2(1+r)} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}r^{n-1}}{n+2} \qquad C_{Ae} = -\frac{1}{6}$$

$$C_{Be} = -\frac{r}{2(1+r)^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}r^{n-1}}{n+2} \qquad C_{Be} = +\frac{1}{3}$$

Frame Constants	I = variable	I = constant
1. Carry-over factors	$C_{AB} = -\frac{C_{Bd}}{C_{Be}}$	$C_{AB} = -\frac{1}{2}$
	$C_{BA} = -\frac{C_{Ae}}{C_{Ad}}$	$C_{BA} = -\frac{1}{2}$
2. Stiffness factors	$k_{AA} = \frac{C_{Be}}{C_{Ad}C_{Be}-(C_{Ae})^2}$	$k_{AA} = +4$
	$k_{BB} = \frac{C_{Ad}}{C_{Ad}C_{Be}-(C_{Ae})^2}$	$k_{BB} = +4$
	$k_{AB}=k_{BA} = -\frac{C_{Ae}}{C_{Ad}C_{Be}-(C_{Ae})^2}$	$k_{AB}=k_{BA} = +2$
3. Fixed-end moments $\left(K_o=\frac{EI_o}{L}\right)$	$M^F_{AB} = -K_o\left(k_{AA}\alpha_{Aq} + k_{AB}\alpha_{Bq}\right)$	
	$M_{BA} = -K_o\left(k_{BA}\alpha_{Aq} + k_{BB}\alpha_{Bq}\right)$	

FIG. 11 (continued)

rotations and relative displacements are rarely important for their own sake and little significance is attached in what units they are expressed. However, when the deformation quantities are required, the various constants, used in the slope-deflection equations, must be dimensionally consistent with the results desired. When E is given in psi, I_o = (inches)^4, L - (feet) and M in ft. kips, the flexural rigidity EI_o should be divided by (144,000), to obtain the rotations in radians and the deflections in feet.

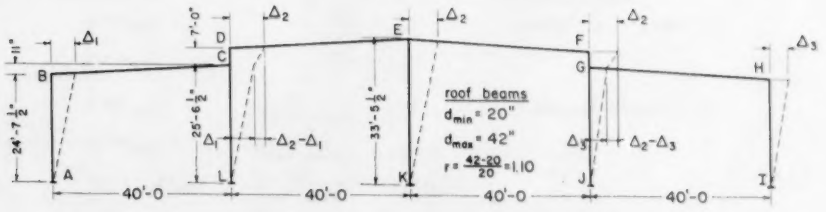
The examples that follow represent actual frames from a large industrial building. Though several of the cases represent design loadings, many others were merely assumed for illustrative purposes. For the purpose of calculating the flexural rigidity the usual practice with reinforced concrete was followed; namely, the reinforcing was neglected and the gross section used in computing the value of the moment of inertia, and the modulus of elasticity taken as $1000f'_c$.

Example 2

A four-bay lean-to frame with tapered roof members is shown in Fig. 12. The direction of the assumed deflections are indicated in the figure. Section properties and the calculated constants are shown tabulated. Slope deflection and equilibrium equations are given in Fig. 13. Results for 6 loading conditions for the case of column bases assumed fixed are given in Table IV (a) and for the hinged bases in Table IV (b). Additional conditions are included under example 3.

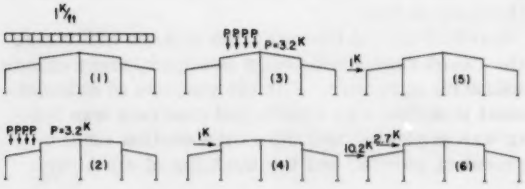
Analysis by Direct Superposition of Simple Beam Rotations (The F.A.S.T.E.R. Method)

There are very few cases in frame analysis, where the magnitude of joint rotation in itself is of any interest. With the slope-deflection method, the restraining moments acting on the end sections of the beam are expressed in terms of the end rotations and relative displacement of supports. From



$E = 3 \times 10^6$ psi

Member	L (ft)	b (in)	d_{min} (in)	I_o (in)^4	$K = \frac{E I_o}{(1,044,000)}$	k_{AA}	k_{AB}	k_{BB}	$C_1 = K_o k_{AA}$	$C_2 = K_o k_{AB}$	$C_3 = K_o k_{BB}$	$C_4 = \frac{C_1 + C_2}{L}$	$C_5 = \frac{C_2 + C_3}{L}$
AB, IH	24.625	20	20	13.333	11280	4.0	2.0	4.0	45120	22560	45120	2749	2749
LC, JG	25.542	20	20	13.333	10870	4.0	2.0	4.0	43500	21750	43500	2555	2555
CD, GF	7.0	20	20	13.333	39680	4.0	2.0	4.0	158730	79365	158730	34020	34020
KE	33.458	20	20	13.333	8300	4.0	2.0	4.0	33210	16605	33210	1489	1489
BC, DE	40.16	18	20	12.000	6225	7.138	6.171	21.778	44435	38407	135540	2063	4332
EF, GH	40.16	18	20	12.000	6225	21.778	6.171	7.138	135540	38407	44435	4332	2063



Note. Tapered beam coefficients for r=1.10 are given in table VI.

FIG. 12

No.	Unstrained EQUATIONS	Load constants case 1	case 2	case 3	case 4	case 5	case 6
0	$M_{AB} - 45.12\,\theta_A - 22.56\,\theta_B + 2.75\,\Delta_1 =$						
1	$M_{BA} - 22.56\,\theta_A - 45.12\,\theta_B + 2.75\,\Delta_1 =$						
2	$M_{BC} - 44.44\,\theta_B - 38.41\,\theta_C =$	- 81.04	- 30.61				
3	$M_{CB} - 38.41\,\theta_B - 135.54\,\theta_C =$	+200.96	+ 77.58				
4	$M_{LC} - 43.50\,\theta_L - 21.75\,\theta_C + 2.56\,\Delta_1 =$						
5	$M_{CL} - 21.75\,\theta_L - 43.50\,\theta_C + 2.56\,\Delta_1 =$						
6	$M_{CD} - 158.73\,\theta_C - 79.37\,\theta_D + 34.02\,\Delta_2 - 34.02\,\Delta_1 =$						
7	$M_{DC} - 79.37\,\theta_C - 158.73\,\theta_D + 34.02\,\Delta_2 - 34.02\,\Delta_1 =$						
8	$M_{DE} - 44.44\,\theta_D - 38.41\,\theta_E =$	- 81.04		- 30.61			
9	$M_{ED} - 38.41\,\theta_D - 135.54\,\theta_E =$	+200.96		+ 77.58			
10	$M_{KE} - 33.21\,\theta_K - 16.61\,\theta_E + 1.49\,\Delta_2 =$						
11	$M_{EK} - 16.61\,\theta_K - 33.21\,\theta_E + 1.49\,\Delta_2 =$						
12	$M_{EF} - 135.54\,\theta_E - 38.41\,\theta_F =$	- 200.96					
13	$M_{FE} - 38.41\,\theta_E - 44.44\,\theta_F =$	+ 81.04					
14	$M_{JG} - 43.50\,\theta_J - 21.75\,\theta_G + 2.56\,\Delta_3 =$						
15	$M_{GJ} - 21.75\,\theta_J - 43.50\,\theta_G + 2.56\,\Delta_3 =$				(Note: Blank spaces denote zeros)		
16	$M_{GF} - 158.73\,\theta_G - 79.37\,\theta_F + 34.02\,\Delta_2 - 34.02\,\Delta_3 =$						
17	$M_{FG} - 79.37\,\theta_G - 158.73\,\theta_F + 34.02\,\Delta_2 - 34.02\,\Delta_3 =$						
18	$M_{GH} - 135.54\,\theta_G - 38.41\,\theta_H =$	- 200.96					
19	$M_{HG} - 38.41\,\theta_G - 44.44\,\theta_H =$	+ 81.04					
20	$M_{IH} - 45.12\,\theta_I - 22.56\,\theta_H + 2.75\,\Delta_3 =$						
21	$M_{HI} - 22.56\,\theta_I - 45.12\,\theta_H + 2.75\,\Delta_3 =$						
22	$M_{BA} + M_{BC} =$						
23	$M_{DC} + M_{DE} =$						
24	$M_{FE} + M_{FG} =$						
25	$M_{HG} + M_{HI} =$						
26	$M_{CB} + M_{CD} + M_{CL} =$						
27	$M_{ED} + M_{EF} + M_{EK} =$						
28	$M_{GF} + M_{GJ} + M_{GH} =$						
29	$M_{AB} + M_{BA} - 24.625\,H_{AB} =$						
30	$M_{LC} + M_{CL} - 25.542\,H_{LC} =$						
31	$M_{KE} + M_{EK} - 33.458\,H_{KE} =$						
32	$M_{JG} + M_{GJ} - 25.542\,H_{JG} =$						
33	$M_{IH} + M_{HI} - 24.625\,H_{IH} =$						
34	$H_{AB} + H_{LC} + H_{KE} + H_{JG} + H_{IH} =$				- 1.0	- 1.0	-12.9
35	$7H_{AB} + 7H_{LC} - M_{CD} - M_{DC} =$					- 7.0	-71.4
36	$7H_{JG} + 7H_{IH} - M_{GF} - M_{FG} =$						
37	$40V_{AB} - 25.542H_{AB} + M_{AB} + M_{CB} =$	+ 800	+256			+ 0.917	+ 9.35
38	$80V_{AB} + 40V_{LC} - 33.458H_{AB} - 33.458H_{LC} + M_{AB} + M_{LC} + M_{ED} =$	+3200	+768	+ 256	+ 0.917	+ 8.833	+92.58
39	$-40V_{IH} - 25.542H_{IH} + M_{IH} + M_{GH} =$	- 800					
40	$-80V_{IH} - 40V_{JG} - 33.458H_{IH} - 33.458H_{JG} + M_{IH} + M_{JG} + M_{EF} =$	-3200					
41	$V_{AB} + V_{LC} + V_{KE} + V_{JG} + V_{IH} =$	+ 160	+ 12.8	+ 12.8			

End conditons
(a) For column bases fixed set $\theta_A, \theta_L, \theta_K, \theta_J$, and θ_I equal to zero.
(b) For column bases hinged set $M_{AB}, M_{LC}, M_{KE}, M_{JG}$ and M_{IH} equal to zero.

FIG. 13

TABLE IV

	(a) COLUMN BASES ASSUMED FIXED						(b) COLUMN BASES ASSUMED HINGED					
	CASE 1	CASE 2	CASE 3	CASE 4	CASE 5	CASE 6	CASE 1	CASE 2	CASE 3	CASE 4	CASE 5	CASE 6
MOMENTS (ft. kip)												
M_{AB}	+ 26.23	+ 8.84	+ 1.86	− 3.01	− 3.07	− 39.43	0.00	0.00	0.00	0.00	0.00	0.00
M_{BC}	− 28.82	− 14.67	+ 5.24	− 2.99	− 3.92	− 48.01	0.00	0.00	0.00	0.00	0.00	0.00
M_{KE}	+ 0.00	+ 1.56	− 6.86	− 2.34	− 1.91	− 25.76	0.00	0.00	0.00	0.00	0.00	0.00
M_{JG}	+ 28.82	+ 1.23	+ 0.49	− 2.99	− 2.61	− 34.72	0.00	0.00	0.00	0.00	0.00	0.00
M_{IH}	− 26.23	+ 1.32	− 0.52	− 3.01	− 2.61	− 34.75	0.00	0.00	0.00	0.00	0.00	0.00
M_{BA}	+ 58.49	+ 22.33	+ 0.12	− 2.28	− 2.00	− 26.53	+ 52.59	+ 19.94	− 0.29	− 4.90	− 4.52	− 59.31
M_{BC}	− 58.49	− 22.33	− 0.12	+ 2.28	+ 2.00	+ 26.53	− 52.59	− 19.94	+ 0.29	+ 4.90	+ 4.52	+ 59.31
M_{CB}	+111.22	+ 35.99	+ 8.76	+ 4.25	+ 1.42	+ 25.91	+103.45	+ 31.82	+ 11.42	+ 7.54	+ 4.18	+ 62.96
M_{CL}	− 52.03	− 25.03	+ 7.12	− 2.51	− 3.98	− 47.38	− 42.62	− 21.15	+ 6.06	− 6.09	− 7.24	− 90.28
M_{CD}	− 59.19	− 10.96	− 15.88	− 1.74	+ 2.56	+ 21.46	− 60.83	− 10.66	− 17.48	− 1.45	+ 3.06	+ 27.31
M_{DC}	+ 61.11	+ 8.94	+ 19.83	− 1.27	+ 0.83	+ 5.04	+ 64.10	+ 10.55	+ 19.06	− 1.61	+ 0.67	+ 2.48
M_{DE}	− 61.11	− 8.94	− 19.83	+ 1.27	− 0.83	− 5.04	− 64.10	− 10.55	− 19.06	+ 1.61	− 0.67	− 2.48
M_{ED}	+218.19	− 4.84	+ 52.07	+ 1.17	+ 0.01	+ 3.25	+215.60	− 5.29	+ 50.91	+ 2.09	+ 0.89	+ 14.75
M_{EK}	0.00	+ 2.03	− 12.51	− 2.33	− 1.79	− 24.52	+ 0.00	+ 0.92	− 9.21	− 4.19	− 3.80	− 50.11
M_{EF}	−218.19	+ 2.81	− 39.56	+ 1.17	+ 1.78	+ 21.27	−215.60	+ 4.37	− 41.70	+ 2.09	+ 2.91	+ 35.36
M_{FE}	+ 61.11	− 0.08	+ 5.48	− 1.27	− 1.21	+ 15.81	+ 64.10	− 0.62	+ 5.84	− 1.61	− 1.67	− 21.35
M_{FG}	− 61.11	+ 0.08	− 5.48	+ 1.27	+ 1.21	− 15.81	− 64.10	+ 0.62	− 5.84	+ 1.61	+ 1.67	+ 21.35
M_{GF}	+ 59.19	+ 1.19	− 5.38	− 1.74	− 1.41	− 19.08	+ 60.83	+ 0.56	− 5.49	− 1.45	− 1.27	− 16.86
M_{GJ}	+ 52.03	+ 0.96	+ 1.23	− 2.51	− 2.21	− 29.27	+ 42.62	+ 0.22	+ 1.60	− 6.09	− 5.86	− 76.21
M_{GH}	−111.22	+ 2.15	+ 4.15	+ 4.25	+ 3.62	+ 48.35	−103.45	+ 0.34	+ 3.89	+ 7.54	+ 7.13	+ 93.07
M_{HG}	+ 58.49	− 1.03	+ 0.79	+ 2.28	+ 1.97	+ 26.29	+ 52.59	+ 0.01	+ 0.32	+ 4.90	+ 4.68	+ 60.96
M_{HI}	− 58.49	+ 1.03	− 0.79	− 2.28	− 1.97	− 26.29	− 52.59	− 0.01	− 0.32	− 4.90	− 4.68	− 60.96
SHEARS (kips)												
V_{AB}	+ 18.76	+ 6.09	− 0.21	− 0.17	− 0.07	− 1.14	+ 18.78	+ 6.12	− 0.29	− 0.32	− 0.20	− 2.88
V_{LC}	+ 37.32	+ 7.05	+ 5.82	+ 0.12	+ 0.10	+ 1.33	+ 37.45	+ 7.07	+ 5.90	+ 0.24	+ 0.21	+ 2.73
V_{KE}	+ 47.84	+ 0.40	+ 8.32	− 0.02	− 0.11	− 1.24	+ 47.55	+ 0.52	+ 8.38	− 0.02	− 0.13	− 1.40
V_{JG}	+ 37.32	+ 0.15	− 1.25	− 0.10	− 0.06	− 0.88	+ 37.45	+ 0.13	− 1.30	− 0.21	− 0.18	− 2.36
V_{IH}	+ 18.76	− 0.08	+ 0.12	+ 0.17	+ 0.14	+ 1.92	+ 18.78	− 0.01	+ 0.11	− 0.32	+ 0.30	+ 3.91
H_{AB}	+ 3.44	+ 1.27	+ 0.08	− 0.21	− 0.21	− 2.68	+ 2.14	+ 0.81	− 0.01	− 0.20	− 0.18	− 2.41
H_{LC}	− 3.17	+ 1.55	+ 0.48	− 0.22	− 0.31	− 3.73	− 1.67	− 0.83	+ 0.24	− 0.24	− 0.28	− 3.53
H_{KE}	+ 0.00	+ 0.11	− 0.58	− 0.14	− 0.11	− 1.50	+ 0.00	+ 0.03	− 0.28	− 0.13	− 0.11	− 1.50
H_{JG}	+ 3.17	+ 0.09	+ 0.07	− 0.22	− 0.19	− 2.51	+ 1.67	− 0.01	+ 0.06	− 0.24	− 0.23	− 2.98
H_{IH}	− 3.44	+ 0.10	− 0.05	− 0.21	− 0.19	− 2.48	− 2.14	− 0.00	− 0.01	− 0.20	− 0.19	− 2.48
ROTATIONS (radians)												
1000 θ_A	0.00	0.00	0.00	0.00	0.00	0.00	− 0.660	− 0.179	− 0.075	+ 0.300	+ 0.300	+ 3.865
1000 θ_L	0.00	0.00	0.00	0.00	0.00	0.00	+ 0.766	+ 0.435	− 0.169	+ 0.312	+ 0.335	+ 4.265
1000 θ_K	0.00	0.00	0.00	0.00	0.00	0.00	+ 0.000	− 0.018	+ 0.197	+ 0.259	+ 0.244	+ 3.184
1000 θ_J	0.00	0.00	0.00	0.00	0.00	0.00	− 0.766	+ 0.007	− 0.038	+ 0.312	+ 0.300	+ 3.898
1000 θ_I	0.00	0.00	0.00	0.00	0.00	0.00	− 0.660	+ 0.004	− 0.009	+ 0.300	+ 0.287	+ 3.733
1000 θ_B	+ 1.430	+ 0.598	− 0.077	+ 0.032	+ 0.048	+ 0.572	+ 1.671	+ 0.704	− 0.088	+ 0.082	+ 0.099	+ 1.236
1000 θ_C	− 1.067	− 0.476	+ 0.087	+ 0.022	− 0.003	+ 0.029	− 1.193	− 0.537	+ 0.109	+ 0.032	+ 0.003	+ 0.114
1000 θ_D	0.449	+ 0.226	+ 0.537	+ 0.028	− 0.025	− 0.178	+ 0.381	+ 0.270	+ 0.569	+ 0.030	− 0.027	− 0.199
1000 θ_E	0.000	+ 0.028	− 0.340	+ 0.000	+ 0.007	+ 0.074	− 0.000	+ 0.037	− 0.358	+ 0.007	+ 0.014	+ 0.165
1000 θ_F	− 0.449	− 0.026	+ 0.171	+ 0.028	+ 0.021	+ 0.292	− 0.381	− 0.018	+ 0.178	+ 0.030	+ 0.025	+ 0.338
1000 θ_G	+ 1.067	+ 0.012	+ 0.034	+ 0.022	+ 0.019	+ 0.250	+ 1.193	− 0.003	+ 0.035	+ 0.032	+ 0.030	+ 0.395
1000 θ_H	− 1.430	− 0.013	− 0.012	+ 0.032	+ 0.028	+ 0.375	− 1.671	+ 0.003	− 0.023	+ 0.082	+ 0.079	+ 1.031
DEFLECTIONS (feet)												
1000 Δ_1	+ 2.194	+ 1.689	+ 1.313	+ 1.136	+ 1.507	+ 19.038	+ 2.889	+ 2.835	− 1.956	+ 5.592	+ 5.734	+ 73.586
1000 Δ_2	+ 0.00	− 0.737	+ 0.809	+ 1.579	+ 1.359	+ 18.129	− 0.000	+ 0.014	+ 0.395	+ 5.856	+ 5.592	+ 72.853
1000 Δ_3	− 2.194	− 0.584	+ 0.095	+ 1.359	+ 1.181	+ 15.719	− 2.889	+ 0.088	− 0.347	+ 5.592	+ 5.356	+ 69.729

considerations of the end rotations of simple beams, the relation between the imposed rotation and the force systems causing it, may be expressed in simpler form, and considerably less arithmetic is involved in determining the needed constants. This is especially significant for members with variable moment of inertia, which may require a considerable amount of mathematical manipulation before one is ready to write down the equations. To distinguish this approach from the more classical slope-deflection method, it may be termed "F.A.S.T.E.R." which is short for Frame Analysis by Superposition Technique of End Rotations. The principal advantage of the F.A.S.T.E.R. Method is the relative ease with which the end rotation due to all the various effects, i.e. bending moments, shear force, axial loads, relative end displacement and torque can be incorporated into the equations. A shearing distortion is significant in the analysis of frames containing members of high depth to span ratio. All other effects should be superimposed when the deflections are used as a measure of vibration, as in the case of large turbo-generator supports, compressor structures and certain other types of machinery supporting frames. Shearing, axial and torsional strains, as well as the extension of the F.A.S.T.E.R. Method to the space frame, and the formulation of a general program for an automated solution of rigid frames, is the subject of a subsequent paper. It should be obvious that the "F.A.S.T.E.R." Method can be considerably slower if the simultaneous equations are not solved on the digital computer. The purpose of this paper is to draw the attention of the structural engineer to the availability of general purpose mathematical programs and how he can use them to obtain rapidly needed information, thus extending the scope of his knowledge of the pattern of behavior, of the structure he is designing under a variety of possible conditions.

For the case of flexure only, equations 10 and 11 give the total angle of rotation caused by the simultaneous action of the imposed loading and the unknown restraining moments at each end. The rotations are in radians, displacements in inches and L in feet.

$$\theta_{AB} = M_{AB}\phi_{AA} + M_{BA}\phi_{AB} + \frac{\Delta}{12L} + \alpha_{Aq} \tag{10}$$

$$\theta_{BA} = M_{AB}\phi_{AB} + M_{BA}\phi_{BB} + \frac{\Delta}{12L} + \alpha_{Bq} \tag{11}$$

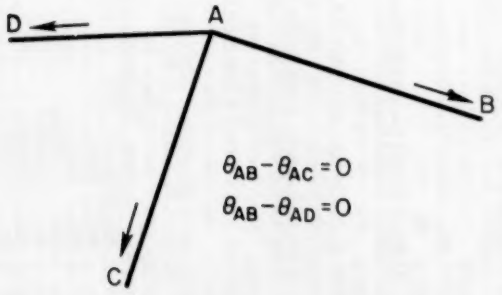


FIG. 14

FIG. 15

NO.	Unstrained EQUATIONS	LOAD CONSTANTS CASE (a)	CASE (b)	CASE (c)	CASE (d)	CASE (e)	CASE (f)	CASE (g)
0	$-\theta_{AB} + 29.551\,M_{AB} - 14.776\,M_{BA} + 33.841\,\Delta_1 = -\measuredangle_{AB} =$	0	- 587.93	0	- 587.93	0	0	- 3249
1	$-\theta_{LC} + 30.665\,M_{LC} - 15.333\,M_{CD} + 32.626\,\Delta_1 = -\measuredangle_{LC} =$	0	0	0	0	0	0	- 2349
2	$-\theta_{KE} + 40.16\,M_{KE} - 20.08\,M_{EK} + 24.907\,\Delta_2 = -\measuredangle_{KE} =$	0	0	0	0	0	0	- 1196
3	$-\theta_{JG} + 30.665\,M_{JG} - 15.333\,M_{GJ} + 32.626\,\Delta_3 = -\measuredangle_{JG} =$	0	0	0	0	0	0	- 783
4	$-\theta_{IH} + 29.551\,M_{IH} - 14.776\,M_{HI} + 33.841\,\Delta_3 = -\measuredangle_{IH} =$	0	0	0	0	0	0	0
	JOINT B $\theta_{BA} - \theta_{BC} = 0$; $M_{BC} = -M_{BA}$							
5	$-14.776\,M_{AB} + 59.355\,M_{BA} + 8.444\,M_{CB} + 33.841\,\Delta_1 = +\measuredangle_{BC} - \measuredangle_{BA} =$	+ 1567	+7761.71	0	+7761.71	0	0	- 3249
	JOINT C $\theta_{CL} - \theta_{CB} = 0$; $\theta_{CL} - \theta_{CD} = 0$ ($M_{BC} = -M_{BA}$)							
6	$-15.333\,M_{LC} + 30.665\,M_{CL} - 8.444\,M_{BA} - 9.769\,M_{CB} + 32.626\,\Delta_1 = +\measuredangle_{CB} - \measuredangle_{CL} =$	- 1016	-4970.36	0	-4970.36	0	0	- 2349
7	$-15.333\,M_{LC} + 30.665\,M_{CL} - 8.401\,M_{CD} + 42\,M_{DC} + 151.674\,\Delta_1 - 119.048\,\Delta_2 = \measuredangle_{CD} - \measuredangle_{CL} =$	0	0	0	0	0	0	- 2349
	JOINT D $\theta_{DC} - \theta_{DE} = 0$; $M_{DE} = -M_{DC}$							
8	$-42\,M_{CD} + 38.205\,M_{DC} + 8.444\,M_{ED} + 119.048\,\Delta_2 - 119.048\,\Delta_1 = +\measuredangle_{DE} - \measuredangle_{DC} =$	0	+7573.30	0	+7573.30	0	0	0
	JOINT E $\theta_{EK} - \theta_{ED} = 0$; $\theta_{EK} - \theta_{EF} = 0$ ($M_{DE} = -M_{DC}$; $M_{FE} = -M_{FG}$)							
9	$-20.08\,M_{KE} + 40.16\,M_{EK} - 8.444\,M_{DC} - 9.769\,M_{ED} + 24.907\,\Delta_2 = +\measuredangle_{ED} - \measuredangle_{EK} =$	0	-4970.36	0	-4970.36	0	0	- 1196
10	$-20.08\,M_{KE} + 40.16\,M_{EK} - 9.769\,M_{EF} - 8.444\,M_{FG} + 24.907\,\Delta_2 = +\measuredangle_{EF} - \measuredangle_{EK} =$	+ 1016	+4970.36	0	+4970.36	0	+ 2075	- 1196
	JOINT F $\theta_{FG} - \theta_{FE} = 0$; $M_{FE} = -M_{FG}$							
11	$+8.444\,M_{EF} + 38.205\,M_{FG} - 4.2\,M_{GF} + 119.048\,\Delta_2 - 119.048\,\Delta_3 = +\measuredangle_{FE} - \measuredangle_{FG} =$	- 1567	-7573.30	0	-7573.30	0	+ 2075	0
	JOINT G $\theta_{GJ} - \theta_{GH} = 0$; $\theta_{GJ} - \theta_{GF} = 0$ ($M_{HG} = -M_{HI}$)							
12	$-15.333\,M_{JG} + 30.665\,M_{GJ} - 9.769\,M_{GH} - 8.444\,M_{HI} + 32.626\,\Delta_3 = +\measuredangle_{GH} - \measuredangle_{GJ} =$	0	+4970.36	0	+4970.36	+ 2075	- 2075	- 783
13	$-15.333\,M_{JG} + 30.665\,M_{GJ} - 8.401\,M_{GF} + 4.2\,M_{FG} + 151.674\,\Delta_3 - 119.048\,\Delta_2 = +\measuredangle_{GF} - \measuredangle_{GJ} =$	0	0	0	0	0	0	- 783
	JOINT H $\theta_{HI} - \theta_{HG} = 0$; $M_{HG} = -M_{HI}$							
14	$+8.444\,M_{GH} - 14.776\,M_{IH} + 59.355\,M_{HI} + 33.841\,\Delta_3 = +\measuredangle_{HG} - \measuredangle_{HI} =$	0	-7573.30	0	-7573.30	+ 2075	- 2075	0
15	$M_{CB} + M_{CD} + M_{CL} =$	0	0	0	0	0	0	0
16	$M_{ED} + M_{EF} + M_{EK} =$	0	0	0	0	0	0	0
17	$M_{GF} + M_{GH} + M_{GJ} =$	0	0	0	0	0	0	0
18	$M_{AB} + M_{BA} - 24.625\,H_{AB} =$	0	+ 67.9	0	+ 67.9	0	0	0
19	$M_{LC} + M_{CL} - 25.542\,H_{LC} =$	0	0	0	0	0	0	0
20	$M_{KE} + M_{EK} - 33.458\,H_{KE} =$	0	0	0	0	0	0	0
21	$M_{JG} + M_{GJ} - 25.542\,H_{JG} =$	0	0	0	0	0	0	0
22	$M_{IH} + M_{HI} - 24.625\,H_{IH} =$	0	0	0	0	0	0	0
23	$H_{AB} + H_{LC} + H_{KE} + H_{JG} + H_{IH} =$	0	0	- 129	- 129	0	0	0
24	$4.779\,M_{CD} + 4.779\,M_{DC} + 4.779\,M_{GJ} + 4.779\,M_{FG} + M_{EK} + M_{KE} =$	0	0	- 90.34	- 90.34	0	0	0
25	$7\,H_{IH} + 7\,H_{JG} - M_{FG} - M_{GF} =$	0	0	0	0	0	0	0
26	$40\,V_{AB} - 25.542\,H_{AB} + M_{AB} + M_{CB} =$	+ 256	+1735.5	+ 935	+1744.85	0	0	0
27	$80\,V_{AB} + 40\,V_{LC} - 33.458\,H_{LC} + M_{AB} + M_{LC} + M_{LD} =$	+ 768	+6061.1	+ 92.58	+6153.68	0	0	0
28	$-40\,V_{IH} - 25.542\,H_{IH} + M_{IH} + M_{GH} =$	0	-1274	0	-1274	0	0	0
29	$-80\,V_{IH} - 40\,V_{JG} - 33.458\,H_{IH} - 33.458\,H_{JG} + M_{IH} + M_{IG} + M_{EF} =$	- 256	-5206	0	-5206	0	0	0
30	$V_{AB} + V_{LC} + V_{KE} + V_{JG} + V_{IH} =$	+ 256	+ 275.64	0	+ 275.64	0	0	0

where

θ_{AB} = total rotation at end A
θ_{BA} = total rotation at end B
} flexure only

ϕ_{AA} = rotation at end A due to unit moment applied at end A of a simply supported beam

ϕ_{AB} = rotation at end A due to unit moment applied at end B of a simply supported beam

ϕ_{BB} = rotation at end B due to unit moment applied at end B of a simply supported beam

α_{Aq} = rotation at end A due to applied loading on the simply supported beam

α_{Bq} = rotation at end B due to applied loading on the simply supported beam

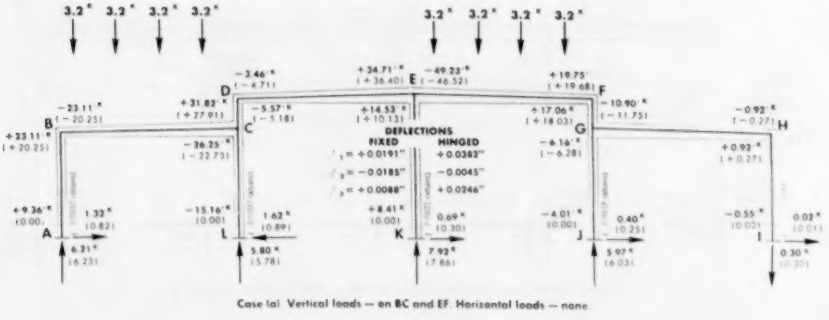
Fig. 14 is a joint in rigid frame, into which three members are shown framing. Neglecting the insignificantly small deformation in the connection itself, the rotations of the end-tangents of each member at A are, obviously, equal, being the same as the rotation of the joint. From the equality

$[\theta_{AB} = \theta_{AC} = \theta_{AD}]$ the rotations of the joints, into which two or more members

TABLE V

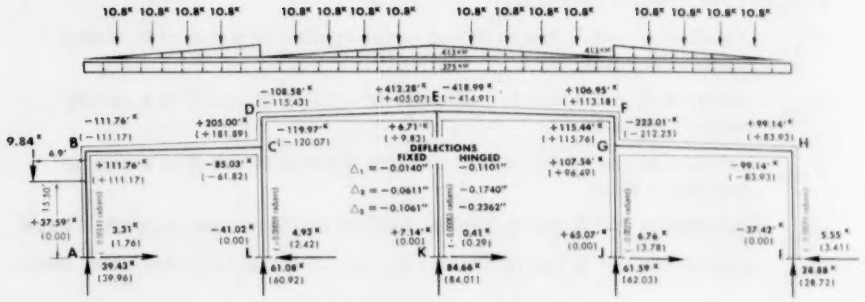
$E = 3 \times 10^6$ psi

Member	L (ft)	$K_B = \frac{E I_B}{L(144000)}$	$\frac{10^6}{K_B}$	C_{Ad}	C_{Ae}	$\frac{C_{Bd}}{C_{Ae}} \times (-1)$	$10^6 \times \phi_{AA}$	$10^6 \times \phi_{BB}$	$10^6 \times \phi_{AB}(-1)$	$\frac{10^6}{12L}$
AB, IH	24.625	11280	88.653	0.3333	0.3333	0.1667	29.551	29.551	14.776	33.841
LC, JG	25.542	10870	91.996	0.3333	0.3333	0.1667	30.665	30.665	15.333	32.626
CD, GF	7.0	39680	25.202	0.3333	0.3333	0.1667	8.401	8.401	4.200	119.048
KE	33.458	8300	120.48	0.3333	0.3333	0.1667	40.160	40.160	20.08	24.907
BC, DE	40.16	6225	160.64	0.1855	0.0608	0.0526	29.804	9.769	8.444	20.75
EF, GH	40.16	6225	160.64	0.0608	0.1855	0.0526	9.769	29.804	8.444	20.75

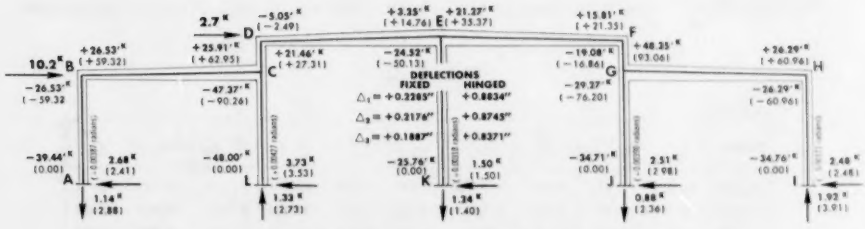


Case (a) Vertical loads — on BC and EF. Horizontal loads — none.

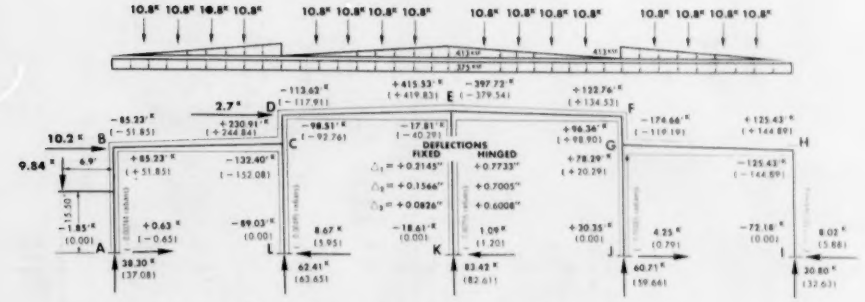
FIG. 16



Case (b) Vertical loads — as shown. Horizontal loads — none.

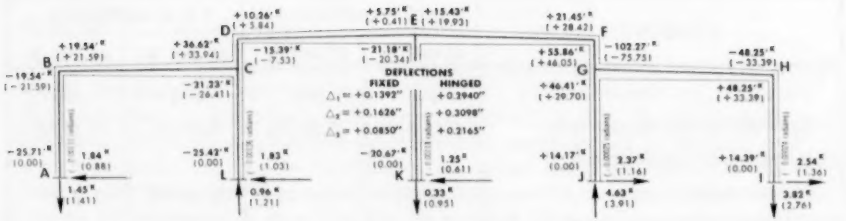


Case (c) Vertical loads — none. Horizontal loads — applied at joints B & D

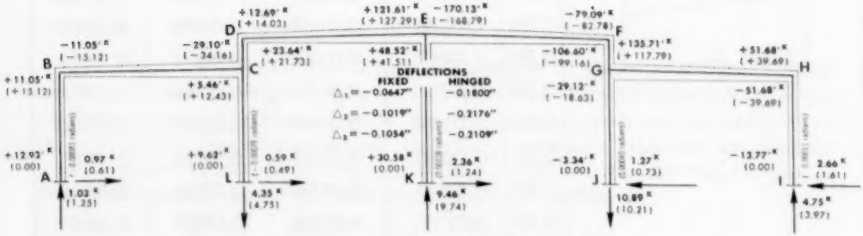


Case (d) Vertical loads — as shown. Horizontal loads — applied at B & D. (Case b + Case c)

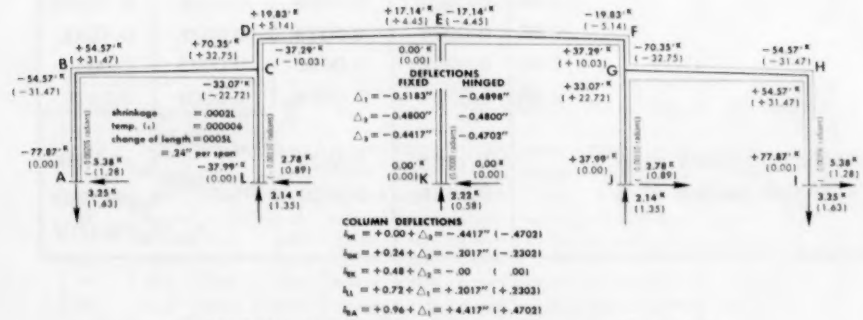
FIG. 16 (continued)



Case (e) Differential settlement. 1" settlement of column HI



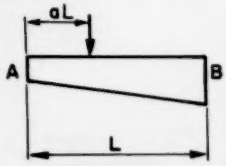
Case (f) Differential settlement. 1" settlement of column GJ



Case (g) Shrinkage plus 50°F. temperature drop (change of length assumed relative to Col. HI)

FIG. 16 (continued)

TABLE VI Tapered beam coefficients ($r = 1.10$)

LOADING		rotation coefficients		F.E.M. coefficients	
		at end A	at end B	at end A	at end B
		(+)	(−)	(−)	(+)
(a) Uniformly distributed load		0.0160	0.0103	0.0511	0.1245
(b) Load increasing uniformly to deeper end		0.0141	0.0102	0.0378	0.1357
(c) Concentrated load distant "aL" from A	a				
	0.05	0.0081	0.0026	0.0418	0.0068
	0.10	0.0142	0.0051	0.0697	0.0238
	0.15	0.0186	0.0074	0.0870	0.0471
	0.20	0.0217	0.0095	0.0961	0.0736
	0.25	0.0237	0.0113	0.0990	0.1009
	0.30	0.0247	0.0129	0.0973	0.1273
	0.35	0.0251	0.0141	0.0922	0.1513
	0.40	0.0248	0.0149	0.0848	0.1720
	0.45	0.0240	0.0155	0.0758	0.1885
	0.50	0.0228	0.0156	0.0660	0.2002
	0.55	0.0212	0.0155	0.0558	0.2067
	0.60	0.0194	0.0150	0.0457	0.2076
	0.65	0.0174	0.0142	0.0361	0.2028
	0.70	0.0151	0.0131	0.0272	0.1920
	0.75	0.0128	0.0117	0.0194	0.1751
	0.80	0.0103	0.0099	0.0126	0.1522
	0.85	0.0078	0.0079	0.0072	0.1232
	0.90	0.0052	0.0055	0.0033	0.0881
	0.95	0.0026	0.0029	0.0008	0.0471
(d) Moment applied at A		+ 0.1855	− 0.0526	$C_{AB} = 0.8644$ $k_{AA} = 7.1384$	
(e) Moment applied at B		− 0.0526	+ 0.0608	$C_{BA} = 0.2833$ $k_{BB} = 21.7784$	
				$k_{AB} = k_{BA} = 6.1707$	

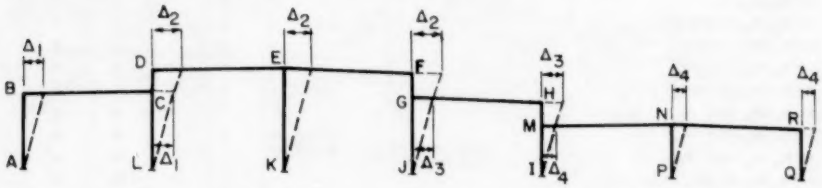
frame, may be eliminated from the equations. The frame of example 2, will be re-analyzed to illustrate the method.

Example 3

The frame of example 2 is solved by the F.A.S.T.E.R. Method. The coefficients are computed and tabulated in Table V. The necessary equations for the seven loading conditions considered are given in Fig. 15. The rotations are given in micro-radians and deflections in centi-inches (1 radian = 10^6 micro-radians; 1 inch = 100 centi-inches). After the coefficients for the unknowns and the constants are arranged in matrix form, compatability with assumed end restraint is provided for, by simply crossing out the end rotations for column bases assumed fixed, or reaction moments for column bases assumed hinged. (See Table III of Example No. 1). The results for both, fixed and hinged bases are shown in Fig. 16. Results in bold face type are for the fixed condition, while the values in parenthesis are for column bases assumed hinged.

Example 4

Two bays are added to the frame of Example 3. The frame and the tabulated coefficients are given in Fig. 17. The necessary equations for the four conditions considered are given in Fig. 18. Fig. 19 presents the results for both, fixed and hinged conditions. Results in bold face type are for column bases assumed fixed, while the values in parenthesis are for column bases assumed hinged. Loading conditions are shown in Fig. 19.



E = 3. 10^6 psi

Member	L (ft)	$K_B \frac{E\,I_B}{L(144,000)}$	$\frac{10^6}{K_{ii}}$	C_{Ad}	C_{Bd}	$\frac{C_{Bd}}{C_{Ad}}=(-1)$	$10^6 \cdot \phi_{AA}$	$10^6 \cdot \phi_{BB}$	$10^6 \cdot \phi_{AB}(-1)$	$\frac{10^4}{12L}$	Relative displmn't. Δ
AB	24.625	11280	88.653	0.3333	0.3333	0.1667	29.551	29.551	14.776	33.841	Δ_1
LC, JG	25.542	10870	91.996	0.3333	0.3333	0.1667	30.665	30.665	15.333	32.626	Δ_1 (LC) Δ_3 (JG)
CD, GF	7.0	39680	25.202	0.3333	0.3333	0.1667	8.401	8.401	4.200	119.048	Δ_2 Δ_2 (CD) Δ_2 Δ_2 (GF)
KE	33.458	8300	120.48	0.3333	0.3333	0.1667	40.160	40.160	20.080	24.907	Δ_2
IM	18.0	15432	64.80	0.3333	0.3333	0.1667	21.600	21.600	10.800	46.296	Δ_3
MH	6.625	41900	23.85	0.3333	0.3333	0.1667	7.950	7.950	3.975	125.786	Δ_3 Δ_4
PN	19.0	13158	76.00	0.3333	0.3333	0.1667	25.333	25.333	12.667	43.860	Δ_4
QR	18.0	13889	72.00	0.3333	0.3333	0.1667	24.000	24.000	12.000	46.296	Δ_4
BC, DE, MN	40.16	6225	160.64	0.1855	0.0608	0.0526	29.804	9.769	8.444	20.750	
EF, GH, NR	40.16	6225	160.64	0.0608	0.1855	0.0526	9.769	29.804	8.444	20.750	

FIG. 17

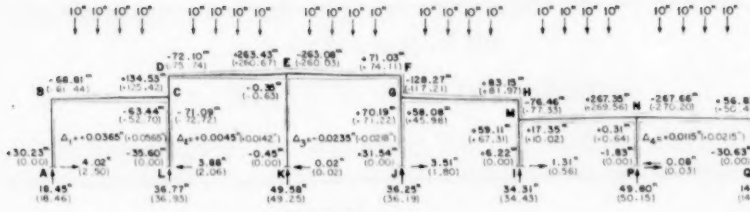
No	Equations	case(a)	case(b)	case(c)	case (d)
0	$-\Theta_A$ 29.551 M_{AB} -14.776 M_{BA} +33.841 Δ_1 = $-\alpha_{AB}$	0	0	0	0
1	$-\Theta_L$ +30.665 M_{LC} -15.333 M_{CL} +32.626 Δ_1 = $-\alpha_{LC}$	0	0	0	+ 783.02
2	$-\Theta_{KE}$ +40.16 M_{KE} -20.08 M_{EK} +24.907 Δ_2 = $-\alpha_{KE}$	0	0	0	+1195.54
3	$-\Theta_J$ +30.665 M_{JG} -15.333 M_{GJ} +32.626 Δ_3 = $-\alpha_{JG}$	0	0	0	+2349.07
4	$-\Theta_M$ +21.6 M_{IM} -10.8 M_{MI} +46.296 Δ_4 = $-\alpha_{IM}$	0	0	0	+4444.42
5	$-\Theta_{PN}$ +25.33 M_{PN} -12.667 M_{NP} +43.86 Δ_4 = $-\alpha_{PN}$	0	0	0	+5263.20
6	$-\Theta_{QR}$ +24 M_{QR} -12 M_{RQ} +46.296 Δ_4 = $-\alpha_{QR}$	0	0	0	+6666.62
7	-14.776 M_{AB} +59.355 M_{BA} + 8.444 M_{CB} + 33.841 Δ_1 = $+\alpha_{BC}-\alpha_{BA}$	+4897	+1567	0	0
8	-15.333 M_{LC} +30.665 M_{CL} - 8.444 M_{BA} - 9.769 M_{CB} +32.626 Δ_1 = $+\alpha_{CB}-\alpha_{CL}$	-3176	-1016	0	+ 783.02
9	-15.333 M_{LC} +30.665 M_{CL} - 8.401 M_{CD} + 4.2 M_{DC} +151.674 Δ_1 +119.048 Δ_2 = $+\alpha_{CD}-\alpha_{CL}$	0	0	0	+ 783.02
10	- 4.2 M_{CD} +38.205 M_{DC} - 8.444 M_{ED} +119.048 Δ_2 -119.048 Δ_1 = $+\alpha_{DE}-\alpha_{DC}$	+4897	0	0	0
11	-20.08 M_{KE} +40.16 M_{EK} - 8.444 M_{DC} - 9.769 M_{ED} + 24.907 Δ_2 = $+\alpha_{ED}-\alpha_{EK}$	-3176	0	0	+1195.54
12	-20.08 M_{KE} +40.16 M_{EK} - 9.769 M_{EF} - 8.444 M_{FG} + 24.907 Δ_2 = $+\alpha_{EF}-\alpha_{EK}$	+3176	+1016	0	+1195.54
13	+ 8.444 M_{EF} +38.205 M_{FG} - 4.2 M_{GF} +119.048 Δ_2 -119.048 Δ_3 = $+\alpha_{FE}-\alpha_{FG}$	-4897	-1567	0	0
14	-15.333 M_{JG} +30.665 M_{GJ} - 9.769 M_{GH} - 8.444 M_{HG} + 32.626 Δ_3 = $+\alpha_{GH}-\alpha_{GJ}$	+3176	0	0	+2349.07
15	-15.333 M_{JG} +30.665 M_{GJ} - 8.401 M_{GF} + 4.2 M_{FG} -119.048 Δ_2 +151.674 Δ_3 = $+\alpha_{GF}-\alpha_{GJ}$	0	0	0	+2349.07
16	- 3.975 M_{HG} + 8.444 M_{GH} +37.754 M_{HM} +125.786 Δ_3 -125.786 Δ_4 = $+\alpha_{HG}-\alpha_{HM}$	-4897	0	0	0
17	-10.8 M_{IM} + 21.6 M_{MI} - 7.95 M_{MH} + 3.925 M_{HM} -125.786 Δ_3 +172.082 Δ_4 = $+\alpha_{MH}-\alpha_{MI}$	0	0	0	+4444.42
18	-10.8 M_{IM} + 21.6 M_{MI} - 29.804 M_{MH} + 8.444 M_{NM} + 46.296 Δ_4 = $+\alpha_{MN}-\alpha_{MI}$	+4897	+1726.61	0	+4444.42
19	-12.667 M_{PN} +25.33 M_{NP} - 9.769 M_{NR} - 8.444 M_{RQ} + 43.86 Δ_4 = $+\alpha_{NR}-\alpha_{NP}$	+3176	0	0	+5263.20
20	-12.667 M_{PN} +25.33 M_{NP} + 8.444 M_{MN} - 9.769 M_{NM} + 43.86 Δ_4 = $+\alpha_{NM}-\alpha_{NP}$	-3176	-1169.44	0	+5263.20
21	-12 M_{QR} + 8.444 M_{NR} + 53.804 M_{RQ} + 46.296 Δ_4 = $+\alpha_{RN}-\alpha_{RQ}$	-4897	0	0	+6666.62
22	$M_{CB}+M_{CD}+M_{CL}$	0	0	0	0
23	$M_{ED}+M_{EF}+M_{EK}$	0	0	0	0
24	$M_{GF}+M_{GH}+M_{GJ}$	0	0	0	0
25	$M_{MH}+M_{MN}+M_{MI}$	0	0	0	0
26	$M_{NM}+M_{NR}+M_{NP}$	0	0	0	0
27	$M_{AB}+M_{BA}-24.625\,H_{AB}$	0	0	0	0
28	$M_{LC}+M_{CL}-25.542\,H_{LC}$	0	0	0	0
29	$M_{KE}+M_{EK}-33.458\,H_{KE}$	0	0	0	0
30	$M_{JG}+M_{GJ}-25.542\,H_{JG}$	0	0	0	0
31	$M_{IM}+M_{MI}-18.0\ \ H_{IM}$	0	0	0	0
32	$M_{PN}+M_{NP}-19.0\ \ H_{PN}$	0	0	0	0
33	$M_{QR}+M_{RQ}-18.0\ \ H_{QR}$	0	0	0	0
34	$H_{AB}+H_{LC}+H_{KE}+H_{JG}+H_{IM}+H_{PN}+H_{QR}$	0	0	- 60	0
35	$M_{CD}+M_{DC}+M_{GF}+M_{FG}-7H_{KE}$	0	0	-140	0
36	$7\ \ H_{AB}+7\ \ H_{LC}-M_{DC}-M_{CD}$	0	0	- 70	0
37	$6.625H_{IM}+6.625H_{PN}+6.625\,H_{QR}-M_{HM}-M_{MH}$	0	0	-132.5	0
38	$M_{AB}+M_{CB}+40V_{AB}-25.542H_{AB}$	+ 800	+256	+ 4.59	0
39	$M_{AB}+M_{LC}+M_{ED}+80V_{AB}+40V_{LC}-33.458H_{AB}-33.458H_{LC}$	+3200	+768	+ 88.33	0
40	$M_{AB}+M_{LC}+M_{KE}-M_{FG}+120V_{AB}+80V_{LC}+40V_{KE}-32.542\,H_{AB}-32.542H_{LC}-32.542H_{KE}$	+7200	+1536	+ 65.41	0
41	$M_{QR}+M_{NR}-40V_{QR}-19\,H_{QR}$	- 800	0	+ 4.59	0
42	$M_{QR}+M_{PN}+M_{MN}-80V_{QR}-40V_{PN}-18H_{QR}-18H_{PN}$	-3200	-332.8	- 9.17	0
43	$M_{QR}+M_{PN}+M_{IM}+M_{GH}-120V_{QR}-80V_{PN}-40V_{IM}-25.542\,H_{QR}-25.542H_{PN}-25.542\,H_{IM}$	-7200	-940.8	+ 13.76	0
44	$V_{AB}+V_{LC}+V_{KE}+V_{JG}+V_{IM}+V_{PN}+V_{QR}$	+ 240	+ 40.8	0	0

End conditions

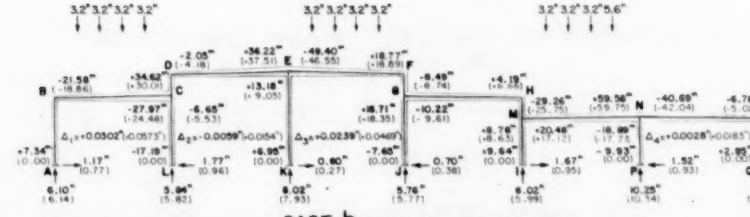
(a) For column bases fixed: set Θ_A, Θ_L, Θ_K, Θ_J, Θ_I, Θ_P and Θ_Q equal to zero

(b) For column bases hinged: set M_{AB}, M_{LC}, M_{KE}, M_{JG}, M_{IM}, M_{PN} and M_{QR} equal to zero

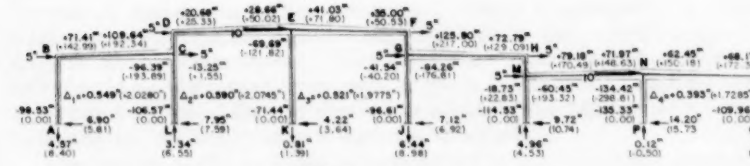
FIG. 18



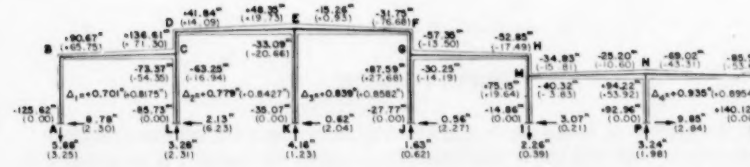
CASE a Vertical loads as shown.
Horizontal loads: none.



CASE b Vertical loads on BC, EF, & MN
Horizontal loads: none.



CASE C Vertical loads: none.
Horizontal loads: as shown.



$\delta_{BA} = -0.00 + \Delta_1 \times +0.701 \quad (+0.8175)$
$\delta_{CL} = -0.24 + \Delta_1 \times +0.461 \quad (+0.5775)$
$\delta_{EK} = -0.48 + \Delta_2 \times +0.299 \quad (+0.3627)$
$\delta_{GJ} = -0.72 + \Delta_3 \times +0.119 \quad (+0.1380)$
$\delta_{MI} = -0.96 + \Delta_3 \times -0.121 \quad (-0.1018)$
$\delta_{NP} = -1.20 + \Delta_4 \times -0.265 \quad (-0.3046)$
$\delta_{RQ} = -1.44 + \Delta_4 \times -0.505 \quad (-0.5446)$

Shrinkage = .0002 L
Temp. (F) = .000006
Change of length = .0005 L
= .24" per span

CASE d Shrinkage plus 50°F temp. drop (Change of length assumed relative to col. BA)

FIG. 19

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ULTIMATE STRENGTH ANALYSIS OF LONG RESTRAINED REINFORCED CONCRETE COLUMNS

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(Proc. Paper 1635)

SYNOPSIS

Theoretical analyses are presented for the ultimate strength of long restrained reinforced concrete columns. The analyses are based on the same principles as the analyses of the ultimate strength of long hinged columns presented in a recent paper;(1) i.e., the treatment of concentrically loaded columns is founded on the tangent modulus theory suggested by Engesser, the treatment of eccentrically loaded columns follows the principles advanced by Karman and both treatments utilize the stress-strain relationship for concrete determined by Hognestad from tests of short eccentrically loaded columns. The consideration of the end restraints requires the addition of a new assumption and necessitates new methods of solution.

The results of the analyses are presented in dimensionless graphs. The following variables are included: (1) concrete strength, (2) percentage of reinforcement, (3) yield point of reinforcement, (4) end restraint, (5) eccentricity, (6) slenderness ratio and (7) duration of loading.

INTRODUCTION

Reinforced concrete columns are usually restrained by adjoining members. At loading of a restrained column, the moment from the load is divided between the column and the restraining members in proportion to their relative

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stiffnesses. An increase of the stiffness of the restraining members decreases the moment acting on the column and increases the ultimate load capacity of the column. Thus the strength of a restrained column depends on both the properties of the column and of the restraining members.

The behavior of a restrained long column under load is illustrated in Fig. 1. The eccentricity of the load is defined as the ratio of the fixed end moment to the axial load. The fixed end moment is resisted in part by the restraining moment of the adjoining members and in part by the moment resistance of the column. The part resisted by the adjoining members is illustrated in Fig. 1 by coil springs. Deflections of the column produce a restraining moment in the spring proportional to the rotation of the end of the column and shown as line 1-3. The moment resistance of the column depends on the cross-section of the column, the stress-strain relationship of the column material and the magnitude of stresses. The sum of the moment resistance and of the restraining moment at midheight of the column is shown as line 1-2; it depends only on the column deflections.

While the moment resistance depends only on the deflection, the external moment depends on the sum of the deflection and eccentricity. The external moment at midheight of the column is shown in Fig. 1 as straight line A-B for eccentricity e and as straight line E-F for eccentricity e_{cr}; the two lines are parallel because their slope depends only on the magnitude of the load P which is taken as a constant. The straight line A-B intersects the curve 1-2 at points C and D for which the sum of the restraining moment and the moment resistance is equal to the fixed end moment. These two points represent two positions of equilibrium, stable at C and unstable at D. When the eccentricity of the load is increased, points C and D approach each other; at the eccentricity e_{cr} the line A-B becomes line E-F, tangent to curve 1-2. At an increase of the eccentricity beyond e_{cr} no position of equilibrium is possible; therefore, the column fails by buckling when load P reaches the eccentricity e_{cr}.

Shorter columns fail by material failure when concrete is crushed in compression. If, for example, curve 1-2 in Fig. 1 ended at point C, the column would fail by material failure when load P reaches the eccentricity e.

The analyses presented herein are limited to columns with equal end eccentricities and equal end restraints.

Notation

The following notation is used throughout the paper:

A	= a parameter given in Table 2
A_s	= total area of longitudinal column reinforcement
B	= a parameter given in Table 2
b	= width of cross-section (Fig. 7)
d	= total depth of cross-section (Fig. 7)
d'	= effective depth of reinforcement (Fig. 7)
d"	= distance between centroids of compression and tension reinforcement (Fig. 7)
e	= end eccentricity of load (Fig. 4)

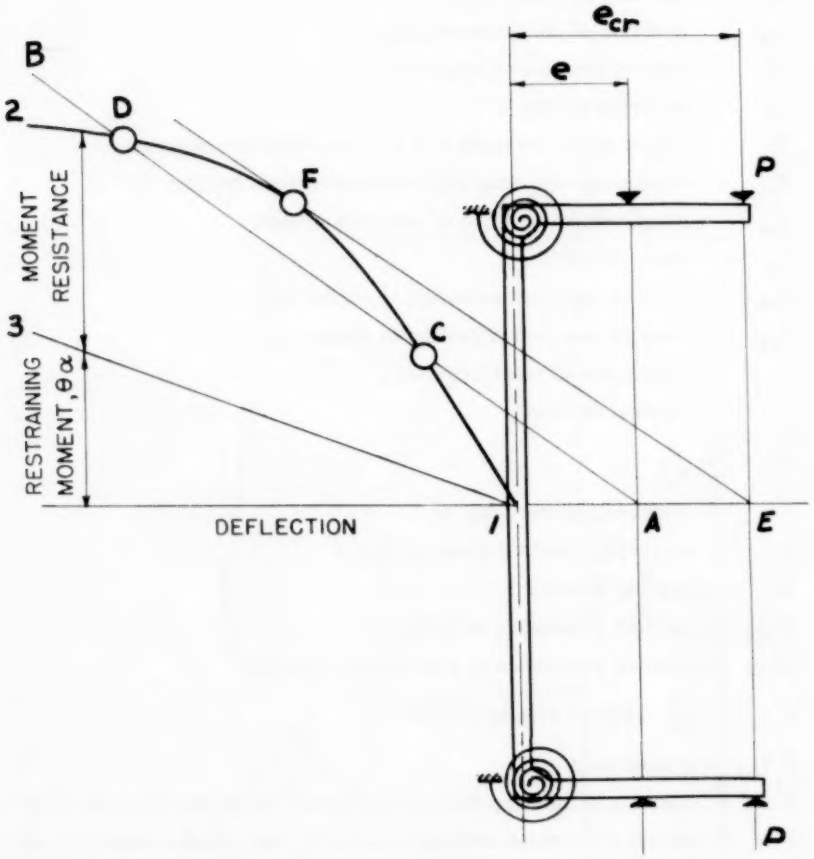

FIG. 1 BUCKLING OF A RESTRAINED COLUMN

e_{cr} = critical eccentricity (Fig. 1)

e_o = initial eccentricity of load

E_c = modulus of elasticity of concrete

E_s = modulus of elasticity of steel

E_t = tangent modulus of elasticity

f_c = concrete stress

f'_c = compressive strength of 6 x 12-in. concrete cylinders

f''_c = compressive strength of concrete in flexure (Fig. 2)

f'_{pr} = compressive strength of concrete prisms

f_s = steel stress

f_{s2} = steel stress corresponding to strain ϵ_2

f_{s3} = steel stress corresponding to strain ϵ_3

f_y = yield point of reinforcement

I = moment of inertia

$k' = \frac{P}{E_t I}$

ℓ = length of column (Fig. 4)

L = equivalent length defined in Fig. 4

M = bending moment

M_{col} = moment resistance of a column

M_{rst} = moment resistance of restraining members

$p = \frac{A_s}{bd}$, ratio of reinforcement

P = applied load

x = distance in the direction of column axis from the middepth (Fig. 4)

y = lateral deflections of column including the initial eccentricity and the effect of end restraint (Fig. 4)

y_m = maximum lateral deflection including the initial eccentricity and the effect of end restraint (Fig. 4)

$\alpha = \frac{M_{rst}}{\theta}$; coefficient of end restraint (Fig. 5)

δ_m = maximum lateral deflection (Fig. 4)

ϵ = strain

ϵ_o = compressive strain corresponding to maximum concrete stress (Fig. 2)

ϵ_1, ϵ_4 = concrete strains (Fig. 7)

ϵ_2, ϵ_3 = steel strains (Fig. 7

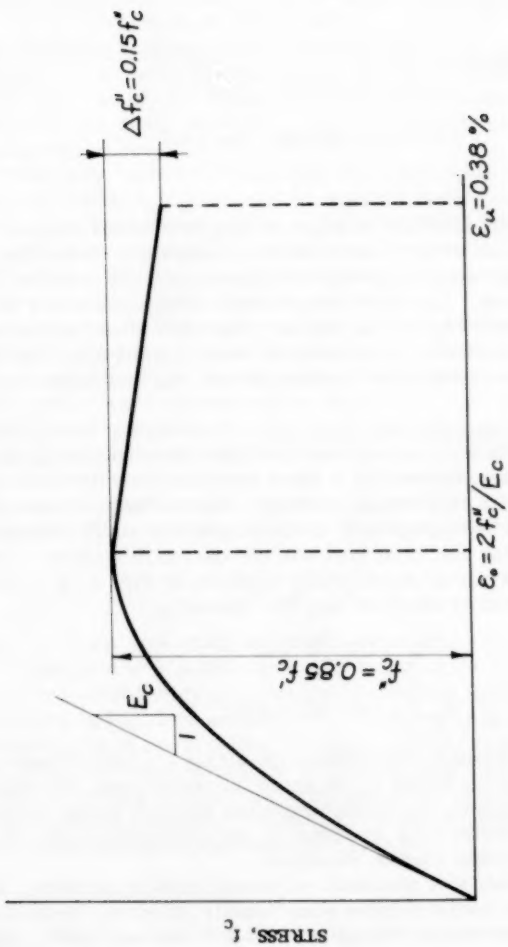

FIG. 2 ASSUMED STRESS-STRAIN RELATIONSHIP FOR CONCRETE

ϵ_u = ultimate concrete strain in flexure (Fig. 2)

ϵ_y = yield point strain of reinforcement

θ = end rotation (Fig. 5)

$\frac{1}{\rho}$ = curvature

$\frac{1}{\rho_m}$ = curvature at x = 0

Ultimate Strength Analysis

Basic Assumptions

The analyses of the ultimate strength of long restrained columns are based on idealized properties of the materials, on assumptions concerning the deformations of the column and on assumptions concerning the restraint offered by the adjoining members. The first two groups of assumptions are the same as those used for the analyses of long hinged columns[1] which analyses were found in good agreement with extensive test data. On the other hand, the third group of assumptions is unique to restrained columns and lacks experimental verification.

Stress-Strain Relationship for Concrete.—The buckling load of a long column depends on the column deflection and can occur at almost any value of strain. It is necessary, therefore, to know the stress-strain relationships for both materials with reasonable accuracy. Accordingly, the stress block for concrete derived by Hognestad[2] from the tests of short columns subjected to combined bending and axial load was chosen for this study.

The chosen stress-strain relationship is shown in Fig. 2. It consists of a parabola characterized by the following two equations,

$$f_c = f''_c\left[\frac{2\epsilon}{\epsilon_o} - \left(\frac{\epsilon}{\epsilon_o}\right)^2\right] \quad (1)$$

$$E_c = 1,800,000 + 460 f''_c \quad , \quad (2)$$

and of a descending straight line between the strain ϵ_o, corresponding to the maximum stress, and the strain ϵ_u at which failure occurs. All other characteristics of the relationship are incorporated in Fig. 2.* It may be noted that for a given cylinder strength of concrete, f'_c, all properties of the idealized stress-strain relationship may be evaluated.

It is assumed further that no tensile stresses exist in concrete. Since the concrete between the cracks carries some tensile stresses, the deflections and end rotations of an actual column are smaller than assumed. The error in deflections results in a decrease but the error in end rotations results in an increase of the ultimate load of the assumed column. Thus, the assumption that no tensile stresses exist in concrete may give either higher or lower ultimate loads than the actual, depending on the relative effects of deflections and end rotations.

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* The relationship $f''_c = 0.85\, f'_c$ is believed to be correct only for vertically cast columns. For horizontally cast columns the relationship $f''_c = f'_c$ was indicated by earlier tests.[3]

Stress-Strain Relationship for Reinforcing Steel.—The assumed stress-strain diagram for steel is shown in Fig. 3. It is representative of steels with a flat yield point used predominantly for reinforcing bars. The modulus of elasticity, E_s, is taken in all calculations as 30,000,000 psi.

Bernoulli's Hypothesis.—Linear distribution of strains across the column cross-section is assumed at all load levels.

Shape of Deflected Column.—It is assumed that the deflected shape of a column is a part of a cosine wave as shown in Fig. 4. This assumption is theoretically correct only for small lateral deflections and for a constant modulus of elasticity along the column. Such conditions are satisfied in concentrically loaded columns.

For eccentrically loaded columns, the use of the cosine wave assumption implies that the modulus of elasticity is constant along the column and equal to the average modulus at the section of failure. The stiffness at the section of failure is the smallest one; therefore, the actual deflections and end rotations are smaller than the computed ones so that the computed ultimate loads may be either lower or higher than the actual.

It is probable that the inaccuracy of the cosine wave assumption has larger effect on the end rotations than on the maximum deflection. However, no quantitative evidence, either experimental or theoretical, is available on the effect of the cosine wave assumption on the accuracy of the computed ultimate load for restrained columns.

End Restraint.—Columns in structures are usually restrained by adjoining members. Rotation at the end of a column produces a restraining moment acting in a direction opposite to that of the rotation. It is assumed in the following analyses that the restraining moment is proportional to the end rotation of the column:

$$M_{rst} = \alpha\theta \qquad (3)$$

where α is the coefficient of end restraint defining the degree of fixity of the adjoining members and θ is the angle of rotation at the end of the column.*

The assumed relationship between the restraining moment and end rotations is shown in Fig. 5 as a solid line. This relationship approximates the actual conditions up to the yielding of tensile reinforcement in the restraining members. After the commencement of yielding, end rotations occur without any appreciable further increase of restraining moment as is indicated in Fig. 5 by the dotted horizontal line. Yielding of the restraining members is not considered in the analyses presented in this paper.

Concentrically Loaded Columns

Buckling.—The buckling load for a concentrically loaded column stressed below the proportional limit of the material was determined theoretically by Euler.[4] For columns stressed beyond the proportional limit, Euler's formulas were modified by Engesser through the use of the tangent modulus of elasticity.[5] Euler's and Engesser's well known solutions are limited to columns with hinged or fixed ends while reinforced concrete columns are usually restrained at their ends by the adjoining frame members. An extensive treatment of buckling of restrained columns may be found in Timoshenko's

* Since θ is a dimensionless quantity, α has the dimension of a moment.

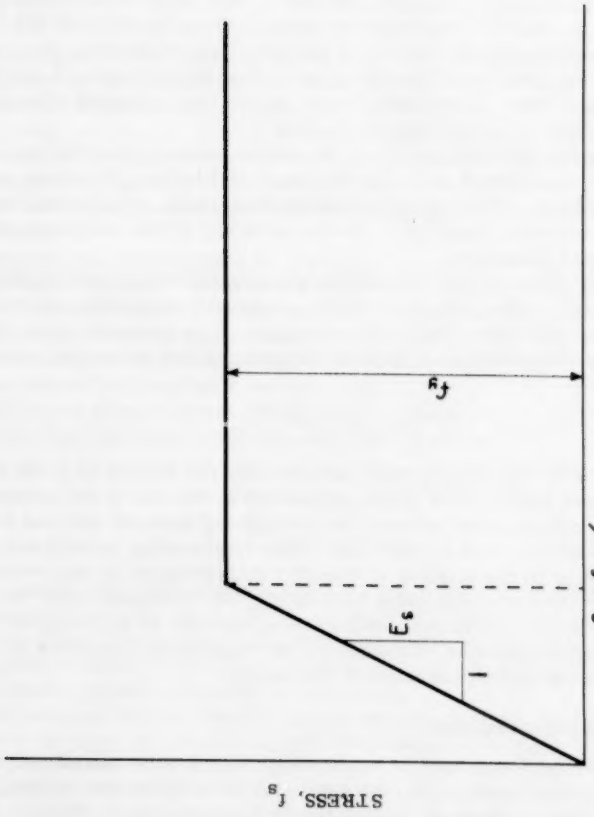

FIG. 3 ASSUMED STRESS-STRAIN RELATIONSHIP FOR REINFORCING STEEL

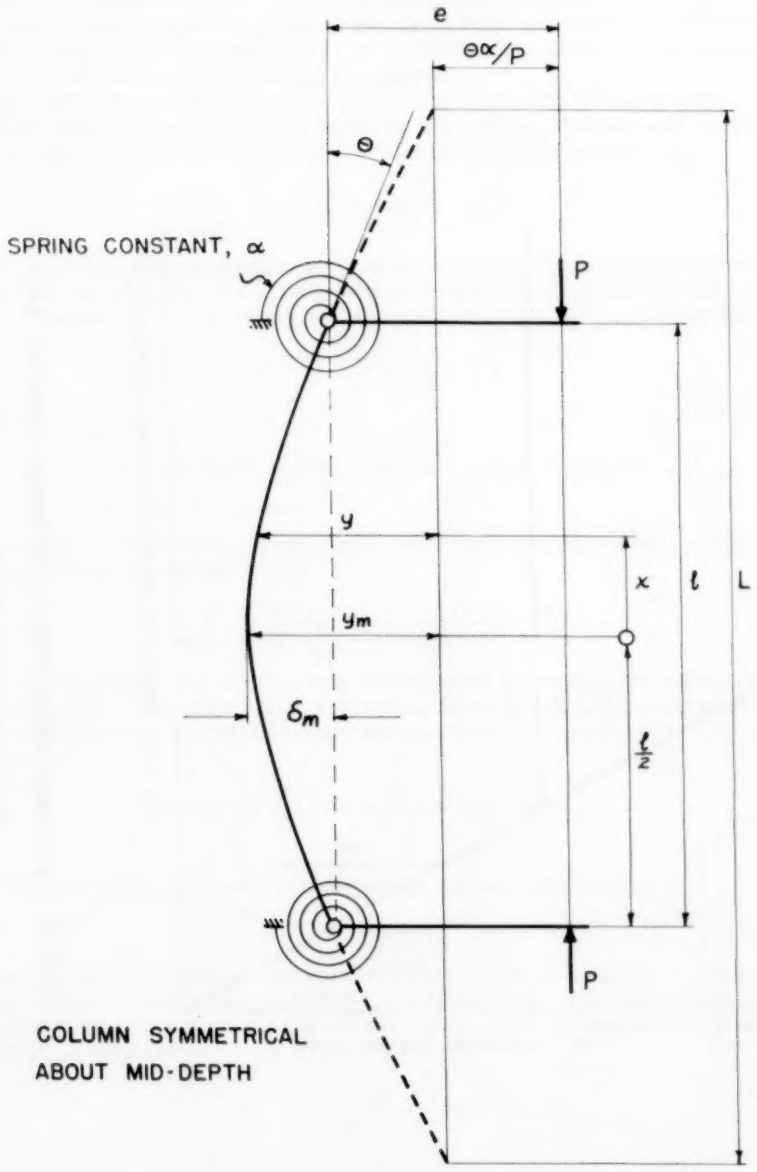

FIG. 4 DEFLECTIONS OF ECCENTRICALLY LOADED RESTRAINED COLUMNS

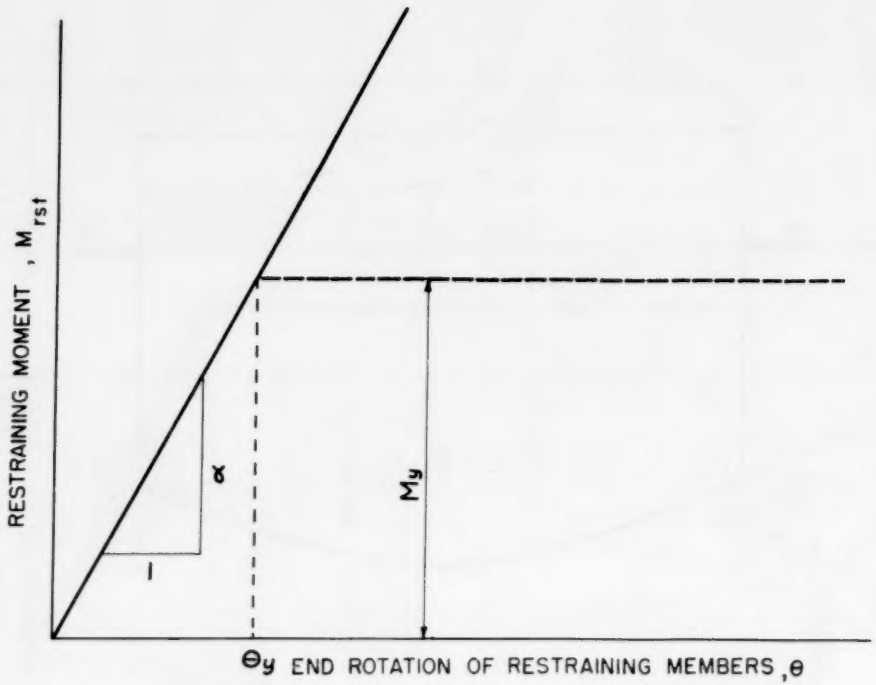

FIG. 5 ASSUMED RELATIONSHIP BETWEEN MOMENT AND ROTATIONS FOR RESTRAINING MEMBERS

Theory of Elastic Stability.[6] Timoshenko's solutions are restricted to columns stressed below the proportional limit; if the stresses exceed the proportional limit, the solutions need to be modified. The modification presented herein utilizes the tangent modulus of elasticity.

The assumed stress-strain diagram for concrete (Fig. 2) has no proportional limit. Accordingly, the following differential equation may be written for small lateral deflections of a reinforced concrete column:

$$-\frac{d^2y}{dx^2} E_t I = M \quad . \tag{4}$$

where E_t is the tangent modulus and I is the moment of inertia of the transformed cross-section. For a hinged column subjected to concentric load P and to moment M applied at one end, the solution of Eq. (4)* may be written as

$$y = \frac{M}{P} \left(\frac{\sin k'x'}{\sin k' \ell} - \frac{x'}{\ell} \right)$$

where $k' = \sqrt{\frac{P}{E_t I}}$

x' = distance from the end at which only load P is applied

ℓ = length of the column.

If equal moments M are applied at both ends, the following solution is obtained by considering the symmetry:

$$y = \frac{M}{P} \left[\frac{\sin k'x'}{\sin k' \ell} + \frac{\sin k' (\ell - x')}{\sin k' \ell} - 1 \right] .$$

The corresponding end rotation may be evaluated as the first derivative of the deflection at $x' = 0$. Expressing P in terms of k', E_t and I, the end rotation of a column subjected to axial load and end moments M is given by the following formula:

$$\theta_M = \left[\frac{dy}{dx'} \right]_{x'=0} = \frac{M}{E_t I} \frac{1}{k'} \operatorname{tg} \left(\frac{k' \ell}{2} \right) .$$

The end rotation of a restrained column may be expressed as

$$\theta = \theta_o + \theta_M$$

where θ_o is the rotation of the corresponding hinged column and θ_M is the end rotation caused by the restraining moments M_{rst}. The restraining moments act in the direction opposite to the direction of the end rotations; thus $M_{rst} = -M$ and

$$\theta = \theta_o - \frac{M_{rst}}{E_t I} \frac{1}{k'} \operatorname{tg} \left(\frac{k' \ell}{2} \right) .$$

Substituting from Eq. (3) for θ and rearranging gives

* The solution is identical to that presented by Timoshenko in Ref. 6, p. 11, except that the tangent modulus E_t has been substituted for the modulus of elasticity E.

$$M_{rst} = \frac{k' E_t I \theta_o}{tg\left(\frac{k' \ell}{2}\right) + \frac{k' \ell}{2} \frac{2E_t I}{\alpha \ell}}$$

At buckling the end rotations θ and thus also M_{rst} approach infinity. This condition is satisfied if

$$tg\left(\frac{k' \ell}{2}\right) + \frac{k' \ell}{2} \frac{2E_t I}{\alpha \ell} = 0 \quad . \tag{5}$$

Eq. (5) is the general expression for the buckling load of a concentrically loaded column with equal elastic restraints at both ends. For a column with both ends fixed $\alpha = \infty$, so that

$$\frac{k' \ell}{2} = \pi$$

and

$$P = \frac{4\pi^2 E_t I}{\ell^2} \quad . \tag{5a}$$

Material Failure.—An inspection of the assumed stress-strain relationship for concrete (Fig. 2) shows that a material failure occurs when the strain reaches the ultimate value for concrete $\epsilon_u = 0.0038$. However, it has been pointed out in a previous paper[1] that failure of a concentrically loaded column may occur at a lower strain and that columns made of commonly used materials usually fail at the concrete strain ϵ_o. Accordingly, it is assumed in the following analysis that a concentrically loaded column fails by material failure if the column strain reaches the value of ϵ_o.

If the steel stress at column failure is designated as f_s, the ultimate load corresponding to material failure may be evaluated from the following equation:

$$\frac{P}{f''_c bd} = \frac{f_s p}{f''_c} + 1 \tag{6}$$

where $f_s = E_s \epsilon_o \leqq fy$.

Evaluation of Ultimate Load.—The tangent modulus E_t and the moment of inertia I in Eq. (5) may be evaluated for any column of known dimensions and properties of materials. The tangent modulus E_t at any strain ϵ (Fig. 2) is equal to $df_c/d\epsilon$. As it is assumed that material failure occurs when ϵ reaches ϵ_o, only the portion of the stress-strain diagram for concrete given by Eq. (1) is of interest for concentrically loaded columns so that

$$E_t = E_c \left(1 - \frac{\epsilon}{\epsilon_o}\right) \tag{7}$$

where E_c is the initial modulus of elasticity given by Eq. (2).

For rectangular symmetrically reinforced concrete columns, the moment of inertia I of the transformed section may be written as:

$$I = \frac{bd^3}{12} + \frac{E_s p}{E_t} \frac{bd\,(d'')^2}{4} \tag{8}$$

where E_s/E_t is the modular ratio; the modular ratio varies with the load. For strains smaller than the yield point strain of the reinforcement ϵ_y, the modulus of elasticity E_s may be taken as 30×10^6 psi. For strains equal and larger than the yield point strain ϵ_y, E_s is equal to zero. Accordingly, the moment of inertia I decreases suddenly when the yield point strain is reached.

For strains smaller than ϵ_o, the relationship between the concentric load and the corresponding strains may be expressed as

$$\frac{P}{f''_c bd} = \frac{f_s p}{f''_c} + \frac{2\epsilon}{\epsilon_o} - \left(\frac{\epsilon}{\epsilon_o}\right)^2 \qquad (9)$$

where f_s is equal to $E_s\epsilon$ for strains smaller than ϵ_y, and equal to f_y for strains equal or larger than ϵ_y.

Eqs. (5), (7), (8) and (9) contain four unknowns: P, E_t, I and ϵ. Thus, the buckling load may be evaluated by simultaneous solution of all four equations. If the resulting ϵ is larger than ϵ_o, the strength of the column is governed by material failure; substitution of $\epsilon = \epsilon_o$ into Eq. (9) results in Eq. (6) for the capacity of columns failing by material failure.

Comparison with Test Data.—No test data are available for concentrically loaded columns partially restrained at the ends. Four long columns with both ends fixed and two long columns with one end fixed and one hinged were tested by Baumann.[7] As all six columns had rectangular cross-section and were reinforced symmetrically, their failure loads may be computed by the procedure presented in this paper.*

The dimensions of all six columns and the properties of materials** are given in Table 1 which includes also the ultimate test and calculated loads, reduced to the dimensionless quantities $P_{test}/f''_c bd$ and $P_{calc}/f''_c bd$, and the ratios P_{test}/P_{calc}. It can be seen that the computed values agree reasonably well with the test values, although for the columns fixed at both ends the calculated values are consistently larger than those obtained from the tests.

Eccentrically Loaded Columns

Buckling.—Kármán[8] proposed a general theory for buckling of eccentrically loaded columns stressed beyond the proportional limit. The theory is based on the stress-strain relationship of the column material and on the assumption of linear strain distribution. Kármán's theory was simplified by Westergaard and Osgood[9] who assumed that the deflected shape of the column is a part of a cosine wave. The modified Kármán's theory was applied to hinged reinforced concrete columns in a previous paper.[1]

The approach used for hinged columns is followed in this paper for restrained columns, except that a rigorous solution for determination of the critical eccentricity is replaced by a graphical procedure illustrated in Fig. 6. The procedure requires computation of several corresponding values of

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* It can be shown that the buckling load for columns fixed at one end and hinged at the other end is given by the following equation:

$$P = \frac{20.16\, E_t I}{\ell^2} \qquad (5b)$$

**Baumann determined the strength of concrete by tests of control prisms; prism strength is comparable to the strength of cylinders.

the eccentricity e and the maximum deflection δ_m for a particular column subjected to a chosen constant load P. The eccentricity e is increased until it has reached e_{cr}, at which value the deflection of the column increases without any further increase in eccentricity.

A deflected restrained column is shown in Fig. 4. It is assumed that the deflected shape of the column is a part of a cosine wave; thus

$$y = y_m \cos \frac{\pi x}{L} \tag{10}$$

The end eccentricities of a restrained column are composed of two parts as shown in Fig. 4: the deflection y at $x = \ell/2$,

$$y_m \cos \frac{\pi \ell}{2L} ,$$

and the eccentricity due to the end restraint,

$$\frac{M_{rst}}{P} .$$

But the restraining moment is assumed proportional to the end rotation θ which may be determined from Eq. (10) as

$$\theta = \left[- \frac{dy}{dx} \right]_{x=\frac{\ell}{2}} = \frac{\pi}{L} y_m \sin \frac{\pi \ell}{2L} .$$

Using Eq. (3), the eccentricity due to the end restraint may be expressed as

$$\frac{\alpha}{P} \frac{\pi}{L} y_m \sin \frac{\pi \ell}{2L}$$

so that the following relationship exists between the end eccentricity e, deflection and the end restraint:

$$e = y_m \cos \frac{\pi \ell}{2L} + \frac{\alpha}{P} \frac{\pi}{L} y_m \sin \frac{\pi \ell}{2L} . \tag{11}$$

It should be noted that for $\theta = 0$, the eccentricity e is equal to the deflection y at $x = \ell/2$:

$$e_o = y_m \cos \frac{\pi \ell}{2L} . \tag{12}$$

Eq. (12) represents the initial eccentricity before any rotation of the ends and before any redistribution of moments to the adjoining members. Since eccentricity is defined as the ratio of moment to axial force, the initial eccentricity, e_o, for a restrained column is equal to the ratio of the fixed end column moment to the axial force.

Eq. (11) gives the vertical ordinate for Fig. 6; the horizontal ordinate, i.e., the corresponding maximum deflection δ_m of a restrained column may be written with the aid of Fig. 4 as

$$\delta_m = y_m + \frac{\alpha}{P} \frac{\pi}{L} y_m \sin \frac{\pi \ell}{2L} - e . \tag{13}$$

Eq. (11) contains three unknowns: e, y_m and L. The necessary two other

FIG. 6 BUCKLING OF RESTRAINED COLUMN

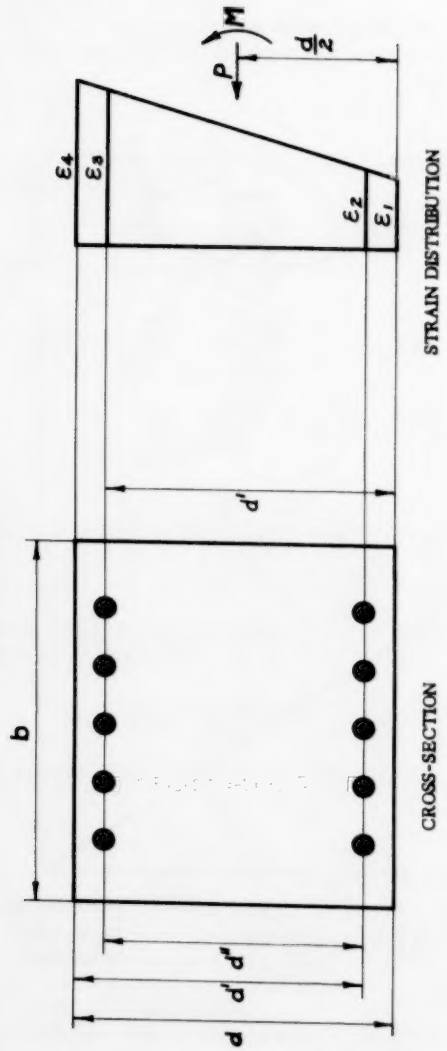

FIG. 7 DISTRIBUTION OF STRAIN IN A COLUMN

relationships may be determined from the column curvature and from the moment equilibrium at column mid-depth.

The curvature and the maximum total deflection are related as follows:

$$\frac{1}{\rho} = -\frac{d^2y}{dx^2} = y_m \left(\frac{\pi}{L}\right)^2 \cos \frac{\pi x}{L} \quad . \tag{14}$$

At the point of maximum moment

$$\frac{1}{\rho_m} = y_m \left(\frac{\pi}{L}\right)^2 \quad .$$

Since the curvature may be expressed also as

$$\frac{1}{\rho_m} = \frac{\epsilon_4 - \epsilon_1}{d} \quad ,$$

the following relationship may be written:

$$\frac{\epsilon_4 - \epsilon_1}{d} = y_m \left(\frac{\pi}{L}\right)^2 \tag{15}$$

where ϵ_4 and ϵ_1 are the strains at the section of maximum moment and d is the depth of the column (Fig. 7).

At the point of maximum moment, the relation between the load, moment and maximum total deflection may be expressed as

$$y_m = \frac{M_{col}}{P} \tag{16}$$

The quantities M_{col} and P may be determined if the strains in the cross-section are known. The relationships between the strains, the load and the moment are discussed in a later part of this paper.

Having Eqs. (11), (13), (15) and (16), the critical eccentricity, for which the assumed load is the buckling load, may be determined as follows. For a constant value of P, several values of $(\epsilon_4 - \epsilon_1)$ are chosen and the corresponding values of M_{col} computed from the relationship between strains, load and moment. For each value of $(\epsilon_4 - \epsilon_1)$ the deflection y_m is determined for Eq. (16) and the value π/L from Eq. (15). For the desired column length ℓ and the end restraint α the values of e and δ_m are then computed from Eqs. (11) and (13). The relationship between the eccentricity e and the deflection δ_m is plotted as shown in Fig. 6. The maximum value, e_{cr}, is the critical eccentricity of the load P.

The procedure derived above represents a general solution for buckling of eccentrically loaded restrained columns. It is applicable to columns of any cross-section and of any material.

Relationships Between Loads, Moments and Strains.—The distribution of stresses in a reinforced concrete column is determined by two strains, the stress-strain relationships for concrete (Fig. 2) and steel (Fig. 3), and the cross-section of the column. The load P and moment M_{col} are expressed in terms of strains by summation of the stresses in the cross-section. The relationships are independent of the end conditions and, therefore, are the same for restrained columns as for hinged columns.(1)

For rectangular cross-sections with symmetrical reinforcement illustrated in Fig. 6, the following equations may be derived:

$$\frac{P}{f''_c bd} = \frac{p(f_{s2} + f_{s3})}{2f''_c} + \frac{A}{\epsilon_4 - \epsilon_1} \tag{17}$$

$$\frac{M_{ccl}}{f''_c bd^2} = \frac{p(f_{s3} - f_{s2})}{4f''_c} \frac{d''}{d} + \frac{B}{(\epsilon_4 - \epsilon_1)^2} - \frac{\epsilon_4 + \epsilon_1}{(\epsilon_4 - \epsilon_1)^2} \frac{A}{2} \quad (18)$$

where parameters A, B and steel stresses f_{s2} and f_{s3} are known functions of concrete strains ϵ_1 and ϵ_4.

Formulas for parameters A and B are listed in Table 2. Four expressions are given for each parameter; each formula is applicable within certain limits for ϵ_4 and ϵ_1. Formulas for cases 1 and 3 refer to full cross-section in compression, while formulas for cases 2 and 4 refer to a cross-section partly in compression and partly in tension.

The steel stresses are related to steel strains ϵ_2 and ϵ_3 as $f_{s2} = E_s \epsilon_2$ and $f_{s3} = E_s \epsilon_3$, and strains ϵ_2 and ϵ_3 may be expressed as

$$\epsilon_2 = \epsilon_4 - (\epsilon_4 - \epsilon_1) \frac{d'}{d} \quad (19)$$

$$\epsilon_3 = \epsilon_1 + (\epsilon_4 - \epsilon_1) \frac{d'}{d} \quad (20)$$

It should be noted from Fig. 3 that the steel stresses f_{s2} and f_{s3} have an upper limit equal to the yield point stress f_y.

In the derivation of Eqs. (17) and (18), compressive strains were taken as positive quantities and tensile strains as negative quantities. The same sign convention should be followed in using these equations.

Material Failure.—An eccentrically loaded column fails by crushing of concrete when the maximum strain ϵ_4 reaches the ultimate value for concrete ϵ_u. The load and moment at failure of such a column may be computed by solving Eqs. (11), (12), (15), (16), (17), (18), (19) and (20) after placing $\epsilon_4 = \epsilon_u$.

Graphs of Ultimate Loads

The procedure for eccentrically loaded columns described in the preceding section does not lend itself easily to a direct evaluation of the buckling load for a particular column. It is more suitable for correlating the ultimate loads to the critical lengths and eccentricities for a whole group of columns having the same properties of the cross-section, the same coefficient of end restraint and the same characteristics of loading other than the magnitude of the initial eccentricity. The ultimate load for a particular column of the group may then be obtained by interpolation.

It is not necessary to make a separate set of calculations for each particular column cross-section. If dimensionless quantities are used instead of the actual loads, moments and dimensions, one group pertains to all columns having the same properties of concrete, properties of steel, percentage of reinforcement, effective depth ratio d'/d, the same coefficient of end restraint and conditions of loading. For convenient use, the results may be tabulated for each group or plotted as is illustrated in Fig. 8-12.

Eleven dimensionless graphs were prepared for rectangular columns with symmetrical reinforcement, equal end restraints and initial eccentricities at both ends. In each graph the quantity $P/f''_c bd$ is plotted as a function of the slenderness ratio ℓ/d for several ratios of the initial eccentricity to the column depth, e_o/d. In all computations the effective depth was taken as $d' = 0.9\,d$.

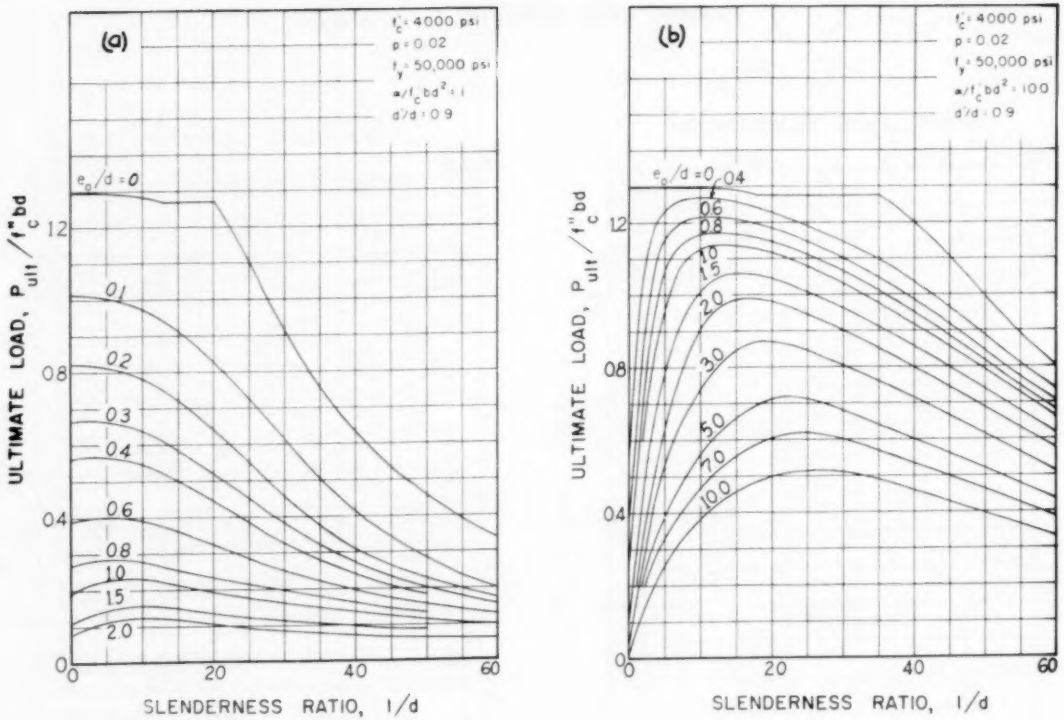

FIG. 8 ULTIMATE LOAD GRAPHS

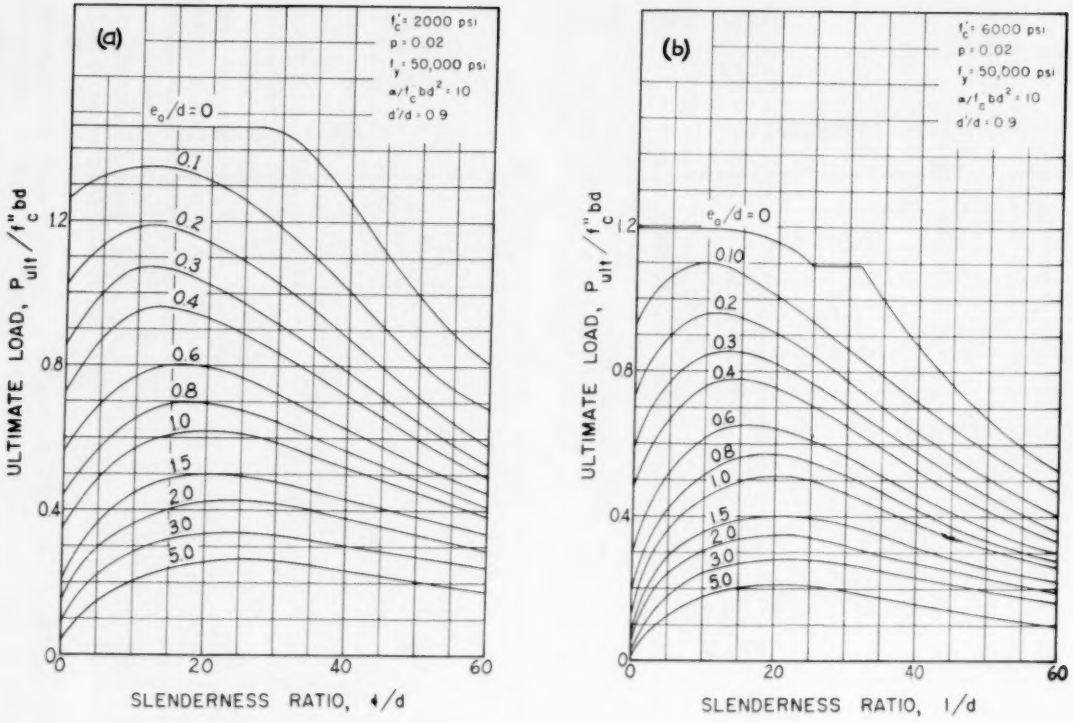

FIG. 9 ULTIMATE LOAD GRAPHS

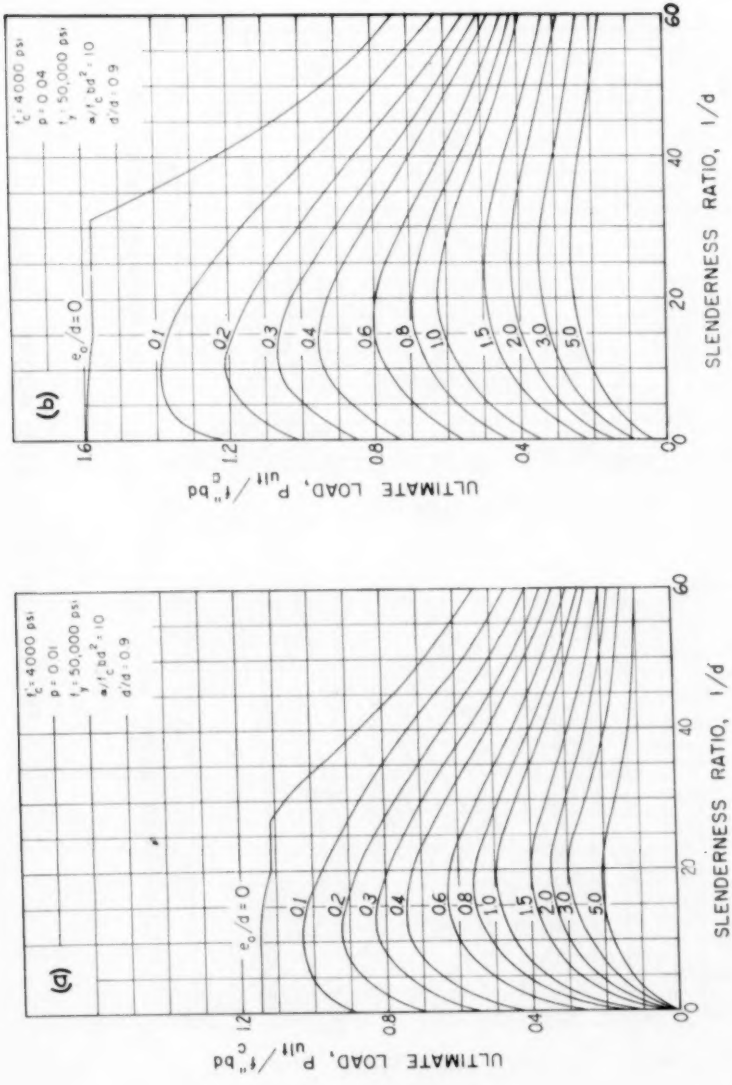

FIG. 10 ULTIMATE LOAD GRAPHS

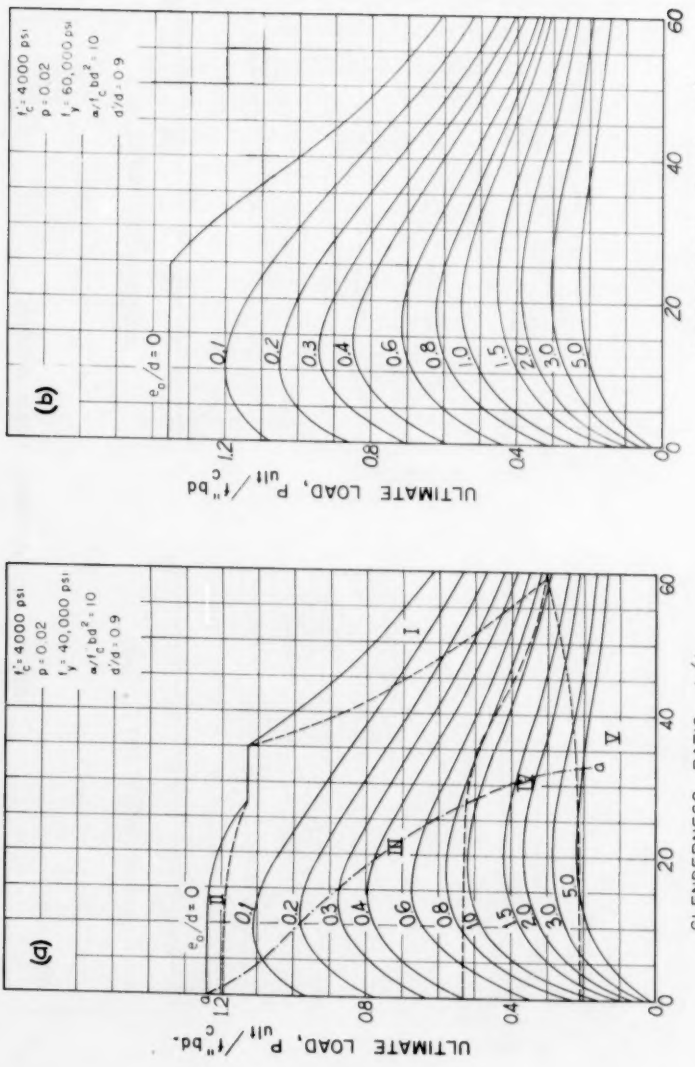

FIG. 11 ULTIMATE LOAD GRAPHS

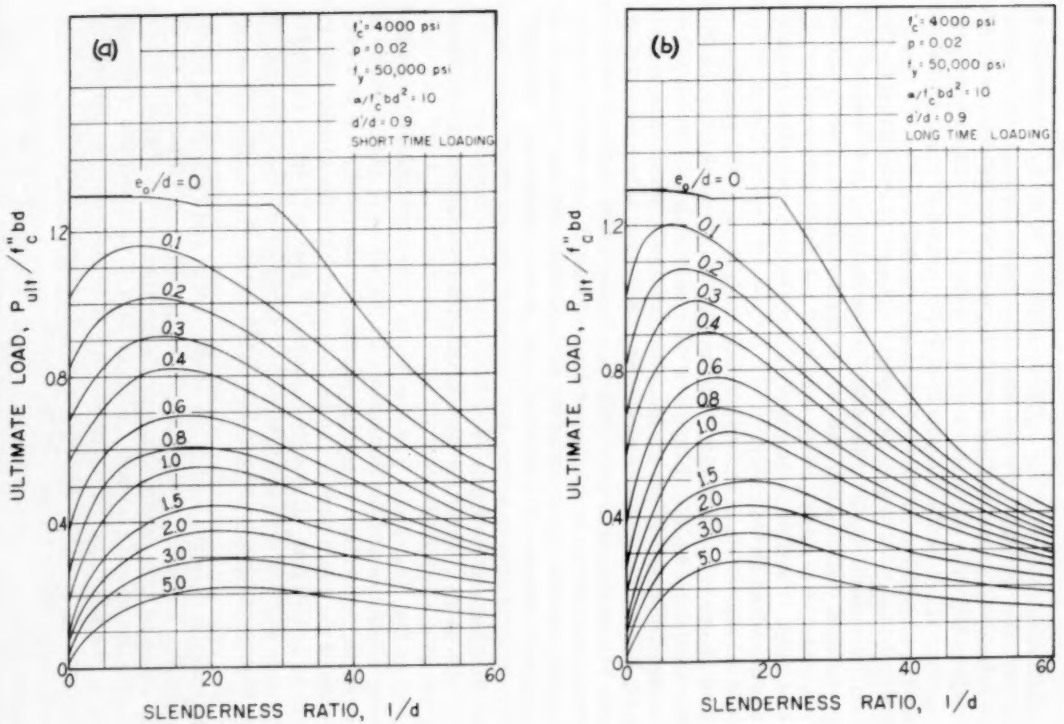

FIG. 12 ULTIMATE LOAD GRAPHS

Discussion of Variables

The ten representative graphs shown in Fig. 8 through 12 illustrate the effects of the slenderness ratio, initial eccentricity ratio, degree of end restraint, concrete strength, percentage of reinforcement, yield point of reinforcement and long duration of loading. An inspection of the graphs reveals that the slenderness ratio ℓ/d, the initial eccentricity ratio e_o/d and the degree of end restraint $\alpha/f'_c\ bd^2$ are the most important variables. It is probable that the relative magnitude of the initial end eccentricities and the relative magnitude of the end restraints at the two ends of the column may have also an important effect on the magnitude of the column load capacity; these effects were not considered in the study reported herein.

Eccentricity of Load.—The graphs in Fig. 8 through 12 are plotted for the initial eccentricity ratios e_o/d. The initial eccentricity e_o is defined as M/P where M is the fixed end moment. As load is applied, a part of the fixed end moment is distributed to the adjoining members and the column carries only part of the moment. Thus the eccentricity at failure of a restrained column is smaller than the initial eccentricity.

The initial eccentricity was chosen for this study because the solution based on such an eccentricity is simpler than that based on the eccentricity at failure. For both solutions the stiffness of the restraining members must be known in advance; fortunately the stiffness of restraining beams is approximately constant and independent of the loading conditions unless the yield point of the tensile reinforcement is exceeded. A solution based on the eccentricity at failure requires also the knowledge of the stiffness of the column, which stiffness varies with the magnitude of the moment and load, and the distribution of moments between all members joined at the column end. The stiffness of the column and the distribution of moments are not required in the solution based on fixed end moments.

It can be seen in all graphs that the ultimate load decreases rapidly with increasing initial eccentricity.

Coefficient of End Restraint.—The coefficient of end restraint α, depends on the loading conditions and the dimensions of the adjoining members, on the concrete strength, percentage of reinforcement, and the yield point of steel. The magnitude of the coefficient α can be estimated from the relationship between moments and strains. It has been shown[10] that $\alpha/f'_c\ bd^2 = 10$ represents reasonably well the average restraint of a reinforced concrete column. Therefore, all graphs except those in Fig. 8 are based on the restraint $\alpha/f'_c\ bd^2 = 10$.

The effect on the ultimate load of the coefficient of end restraint is shown in Fig. 8a and 8b. There is no effect for columns with $\ell/d = 0$ but the effect is large for columns with intermediate and large slenderness ratios. With increasing coefficient of end restraint, a larger part of the fixed end moments is attracted by the adjoining members and only a smaller part by the column; thus the eccentricity at failure decreases with increasing coefficient of end restraint and the ultimate strength is increased.

Slenderness Ratio.—An examination of Fig. 8 through 12 shows that at small slenderness ratios the ultimate load increases with increasing length of the column; however, at large slenderness ratios the ultimate load decreases with increasing length of the column. The increase of ultimate load is caused by the end restraint and the decrease is caused by the column deflections and instability.

The increase at small slenderness is more pronounced for columns with high end restraints. For columns with small end restraints (Fig. 8a) it is very small and for hinged columns[1] it does not exist at all.

Concrete Strength.—It has been pointed out earlier that an eccentrically loaded column can fail by material failure or by buckling. If the maximum compressive strain reaches the ultimate value ϵ_u the column fails by crushing of concrete, i.e., by material failure. If the column becomes unstable before reaching the ultimate strain ϵ_u it fails by buckling. The transition between the material failures and failures by buckling is shown in Fig. 11a as line a-a. Columns to the left of the line a-a fail by crushing of concrete; columns to the right of the line a-a fail by buckling.

Yield Point of Reinforcement.—The presence or absence of yielding at failure is indicated in Fig. 11a. In area I the steel stresses at failure are below the yield point value, in area II all steel yields in compression, in area III the steel yields in compression on one face only, in area IV the steel yields in compression on one face and in tension on the other, and in area V the steel yields in tension on one face only. The yield point has no effect on the ultimate load in area I and has the largest effect on the ultimate load in area IV.

Long Duration of Loading.—The graphs in Fig. 8 through 11 and 12a represent the strength of columns subjected to short-time loading.* Studies of short eccentrically loaded columns[3] and of long hinged columns[1] have indicated a lower ultimate strength when failure occurs under sustained loads. The primary effects of creep appear to be a substantial increase of the maximum concrete strain at failure ϵ_u and a relatively small decrease of the maximum concrete stress f''_c. Neglecting the small effect of creep on concrete strength, the stress-strain relationship for long-time loading may be taken as shown in Fig. 2 but with the strain ordinates multiplied by a factor of 2. The ultimate loads may then be computed in the same manner as for short-time loading; Fig. 12b represents an ultimate load graph computed in this manner. A comparison of this graph with Fig. 12a illustrates the difference between the maximum load which a column can resist for a short time and that which can be sustained indefinitely.

Limitations

In the preceding analyses of the ultimate load of restrained columns, it was assumed that the end rotation of the column is smaller than the rotation which causes formation of a plastic hinge in the adjoining member. It has been assumed that the moment which resists the rotation at the ends of the column increases linearly with the rotation. However, a plastic hinge may form in the restraining members before or at the ultimate column load thus decreasing the strength of the restrained column as is illustrated in Fig. 13. The restraining moment produced by the restraining members increases linearly with the end rotations until the plastic hinge has formed but further end rotations occur without any increase in the restraining moment (curve 1-3). The sum of the restraining moment and moment resistance is indicated by the curve 1-2. Buckling of the column occurs when the maximum deflection of the

* A test to failure in a conventional testing machine carried out in a few hours.

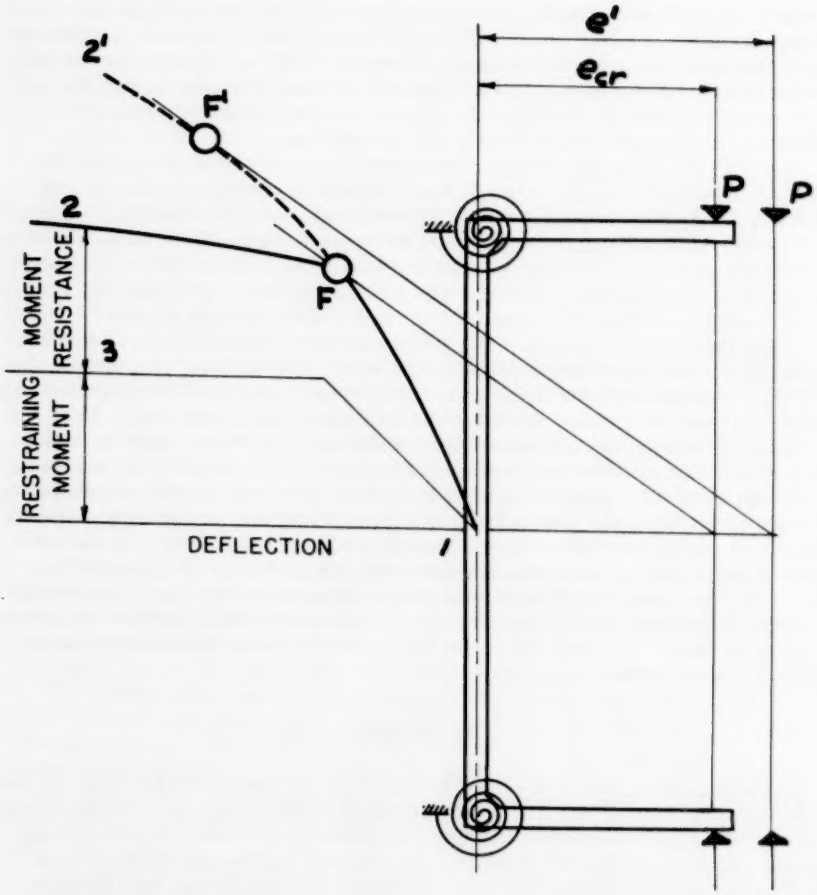

FIG. 13 EFFECT OF PLASTIC HINGE

column has reached the point F and the corresponding eccentricity of the applied load is equal to e_{cr}.

The dashed curve 1-2 in Fig. 13 is the sum of the moment resistance and the restraining moment for the case when a plastic hinge does not form. Buckling of the column occurs at the eccentricity e'. The magnitude of the difference between the critical eccentricities depends on how closely the points F' and F are located.

The ultimate capacity of a column may be decreased considerably by the formation of plastic hinges. Thus the effect of the formation of plastic hinges must be considered if the increase in capacity due to end restraint is utilized in the design.

The theory for the ultimate strength of long hinged columns presented in an earlier paper[1] was verified by an extensive set of test data. On the contrary, except for a few tests of concentrically loaded columns with fixed ends, no test data are available for long restrained reinforced concrete columns. The absence of test data is especially important because the theory involves two assumptions in addition to those used in the theory of hinged columns: (1) for computations of end rotations the deflected shape of the column is assumed a part of the cosine wave, and (2) the restraining moment is assumed proportional to the end rotations. No quantitative evidence is available on the effect of the cosine wave assumption on the accuracy of the computed ultimate load of restrained long columns and on the relationship between restraining moments and end rotations. The effects of these additional assumptions on the ultimate strength of restrained columns should be investigated by experimental means.

ACKNOWLEDGMENT

The studies reported in this paper were developed as a part of a cooperative investigation conducted by the Engineering Experiment Station of the University of Illinois under the auspices and sponsorship of the Reinforced Concrete Research Council of the Engineering Foundation. The paper is based on the doctoral dissertation of the senior author.[10]

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TABLE I

Tests of Concentrically Loaded Fix-Ended Long Columns

All columns had 1 per cent of longitudinal reinforcement. In the calculations of ultimate loads, the yield point of the reinforcement was assumed as 46,600 psi and the effective depth was assumed as d' = 0.9 d.

Specimen	Concrete Strength f'_c,* psi	Cross-Section d x b in. x in.	Slenderness ℓ/d	$\frac{P_{test}}{f''_c\ bd}$	$\frac{P_{calc}}{f''_c\ bd}$	$\frac{P_{test}}{P_{calc}}$
		Both Ends Fixed				
17	3360	3.54 x 7.87	32.4	1.065	1.142	0.93
21	5810	3.50 x 7.87	32.7	0.907	1.036	0.88
22	5810	3.50 x 7.87	32.7	1.029	1.036	0.99
23	6390	5.08 x 9.76	21.4	0.897	1.126	0.80
		One End Fixed, One Hinged				
18	3360	3.58 x 7.91	34.2	0.996	0.940	1.06
20	4070	5.12 x 9.84	23.0	1.094	1.109	0.99

* All columns were cast horizontally; therefore $f'_c = f'_{pr}/0.85$.

TABLE 2

RELATIONSHIP BETWEEN LOAD, MOMENT AND STRAINS FOR SYMMETRICAL RECTANGULAR CROSS-SECTIONS OF REINFORCED CONCRETE

$$\frac{P}{f_c'bd} = \frac{p(f_{s2}+f_{s3})}{2f_c'} + \frac{A}{(\epsilon_4-\epsilon_1)} \qquad \frac{M_{col}}{f_c'bd^2} = \frac{p(f_{s3}-f_{s2})}{4f_c'}\frac{d'}{d} + \frac{B}{(\epsilon_4-\epsilon_1)^2} - \frac{(\epsilon_4+\epsilon_1)}{(\epsilon_4-\epsilon_1)^2}\frac{A}{2}$$

Case 1 $\epsilon_4<\epsilon_o$ $\epsilon_1>0$	$A = \frac{(\epsilon_4^2-\epsilon_1^2)}{\epsilon_o} - \frac{(\epsilon_4^3-\epsilon_1^3)}{3\epsilon_o^2}$ $\qquad$ $B = \frac{2(\epsilon_4^3-\epsilon_1^3)}{3\epsilon_o} - \frac{(\epsilon_4^4-\epsilon_1^4)}{4\epsilon_o^2}$
Case 2 $\epsilon_4<\epsilon_o$ $\epsilon_1<0$	$A = \frac{\epsilon_4^2}{\epsilon_o} - \frac{\epsilon_4^3}{3\epsilon_o^2}$ $\qquad$ $B = \frac{2\epsilon_4^3}{3\epsilon_o} - \frac{\epsilon_4^4}{4\epsilon_o^2}$
Case 3 $\epsilon_o<\epsilon_4<\epsilon_u$ $\epsilon_1>0$	$A = \frac{2\epsilon_o^3-3\epsilon_o\epsilon_1^2+\epsilon_1^3}{3\epsilon_o^2} + \frac{(\epsilon_u-0.85\epsilon_o)(\epsilon_4-\epsilon_o)-0.075(\epsilon_4^2-\epsilon_o^2)}{(\epsilon_u-\epsilon_o)}$ $B = \frac{2(\epsilon_o^3-\epsilon_1^3)}{3\epsilon_o} - \frac{(\epsilon_o^4-\epsilon_1^4)}{4\epsilon_o^2} + \frac{(\epsilon_u-0.85\epsilon_o)(\epsilon_4^2-\epsilon_o^2)-0.10(\epsilon_4^3-\epsilon_o^3)}{2(\epsilon_u-\epsilon_o)}$
Case 4 $\epsilon_o<\epsilon_4<\epsilon_u$ $\epsilon_1<0$	$A = \frac{2}{3}\epsilon_o + \frac{(\epsilon_u-0.85\epsilon_o)(\epsilon_4-\epsilon_o)-0.075(\epsilon_4^2-\epsilon_o^2)}{(\epsilon_u-\epsilon_o)}$; $B = \frac{5\epsilon_o^2}{12} + \frac{(\epsilon_u-0.85\epsilon_o)(\epsilon_4^2-\epsilon_o^2)-0.10(\epsilon_4^3-\epsilon_o^3)}{2(\epsilon_u-\epsilon_o)}$

Note: Compressive strains should be taken as positive quantities, tensile strains as negative quantities. The steel stresses $f_{s2} = E_s\epsilon_2$ and $f_{s3} = E_s\epsilon_3$ have an upper limit equal to the yield point stress f_y

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USE OF MODERN COMPUTERS IN STRUCTURAL ANALYSIS[a]

Ray W. Clough,[1] A.M. ASCE
(Proc. Paper 1636)

INTRODUCTION

Current shortages in the supply of engineering personnel combined with continuing developments and expansion in the computer industry have led many engineering organizations recently to consider whether electronic computers might be used effectively in their particular fields of endeavor. The fact that at least 26 state highway departments as well as numerous private highway engineering consulting offices are now making use of computers demonstrates that they have conclusively proven their worth in this phase of civil engineering. However, with a few important exceptions, the application of electronic computers in the design and analysis of civil engineering structures has been very limited to date. It is the purpose of this presentation to describe briefly how computers may be put to use in structural engineering work.

Fortunately, civil engineers will find that a considerable amount of work has been done in this field already. Electronic computers have been used extensively for the solution of structural problems in the aircraft industry for several years, and much of this experience may be applied directly in civil engineering work. This pioneering activity in the computer field was forced upon the airplane structural design groups by two inherent characteristics of their design problem. First, the basic requirement of minimum weight makes necessary precise calculations of stresses and deflections rather than crude, rule of thumb approximations; and second, because of aerodynamic as well as weight efficiency considerations, the airplane must assume a complex form not amenable to simple idealizations. Even a long slender wing to which the elementary beam theory may be applied with reasonable accuracy is a relatively complex structure, by ordinary civil engineering standards, involving both unsymmetrical bending and shear flow in a multi-cell box. It is apparent,

Note: Discussion open until October 1, 1958. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. Paper 1636 is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 84, No. ST 3, May, 1958.

a. Presented at the Annual Convention of the American Society of Civil Engineers, New York, N. Y., October 13, 1957.

1. Associate Prof. of Civ. Eng., Univ. of California, Berkeley, Calif.

then, that when the wing is shortened and broadened to meet the aerodynamic requirements of supersonic flight, its analysis becomes an exceedingly complex problem. The elastic axis concept no longer is applicable; instead the wing must be treated as a non-uniform two-dimensional plate or as a three-dimensional shell structure, with obvious difficulties in either case.

It was complications such as this which first led airplane structural engineers to consider the use of automatic computers in their work. However, one additional circumstance which contributed to their leadership in the field and should not be overlooked was the fact that electronic computers were available to them, a pleasant situation enjoyed by very few civil engineers at that time. Fortunately, this situation has changed completely in the past two years, and now computer facilities are available to almost anyone having a problem which warrants their use. For this reason, structural engineers should begin now to consider seriously how electronic computers might be applied in their design and analysis work, and it is hoped that this paper may throw some light on that question.

The presentation will be divided generally into three parts. First will be given a brief functional description of an electronic computer, for the benefit of those who have not had an opportunity to work with one. Then some general methods by which computers can be used to solve structural problems will be discussed, and finally there will be presented a brief review of some of the civil engineering structural work which has been done by computers. A rather extensive bibliography on matrix analysis and electronic computer analysis of structures is given at the end of the paper.

Functional Description of a Digital Computer

As is generally known, two distinct types of electronic computers, analogue and digital, are used widely in the solution of engineering problems. The digital computer, as its name implies, deals directly with numbers, manipulating them much in the same way as is done with pencil and paper or a desk calculator. On the other hand, the analogue computer deals with physical quantities, such as voltages or currents, rather than with numbers, and the solution is obtained in terms of an electrical analogue of the mathematical or physical system under consideration. (A slide rule is the best known example of an analogue computer, numbers being represented by distances on the scales, and the calculations being performed by physical manipulation of these distances.) Analogue computers have many uses in solving specialized problems, but because of their greater versatility digital computers have been much more widely used in structural analysis and this discussion will be limited to that type of computer.

The procedure followed in the operation of an automatic computer may be explained most clearly by comparison with a similar calculation carried out on a desk calculator. (Such a comparison is given in the excellent book "Mathematics and Computers" by G. R. Stibitz and J. A. Larrivee, McGraw-Hill, N. Y., 1957, which is to be recommended to any laymen interested in the application of computers to their work.) As an example, suppose that an engineer wishes to determine the height of a mountain peak above a certain point as shown in Fig. 1. From a map he determines that the horizontal distance (d) from the point to the peak is 20,000 ft. and with a transit he measures the vertical angle (θ) to be 15^0. Now drawing upon his college education and

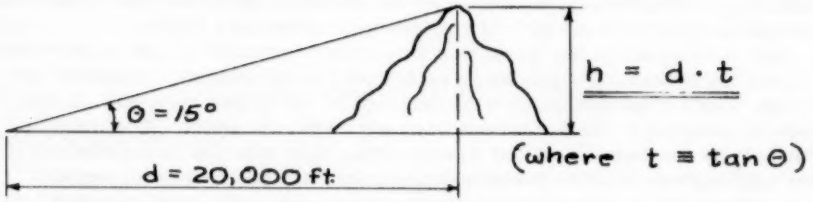

Fig. 1 Formula for Height Calculation

subsequent experience, he recalls that the height (h) will be given by the formula presented in the figure.

In his office, the engineer has a computist who will carry out the calculation on a desk calculator. (The word "computist" is used to designate the human being who carries out the computation, as distinguished from the mechanical "computer".) This computist has very definite limitations in her abilities—she knows only how to operate a desk calculator, and to look up and transcribe numbers. In order that the computist can carry out this required calculation, it is necessary for the engineer to write out a program specifying in complete detail all of the operations she is to perform, and he must also supply her with a work sheet on which the numbers can be recorded. The program and work sheet might look as shown in Fig. 2.

Computist Program

a. From Data Sheet write "d" in space 1
b. From Data Sheet write "θ" in space 2
c. In Table look up value of "t" corresponding to number in space 2 and record in space 3
d. On desk calculator form product of numbers in spaces 1 and 3 and record in space 4
e. Write result from space 4 on separate sheet for delivery to Engineer

Work Sheet

1	20,000
2	15°
3	0.26795
4	5358.5

Data Sheet

d = 20,000 feet
θ = 15 degrees

Fig. 2 Program for Computist Solution

In addition, it will be necessary to supply the data for this particular problem, as shown at the bottom of the figure. With this program and work

sheet, the computist could now work out the height of the mountain, or of any other object for which the data was provided. In fact, it would only be worth while to go to the trouble of preparing the program and work sheet if the same calculation was to be performed with many different sets of data.

Now the important fact about this very simple example is that it represents exactly the procedure which would be followed in an automatic computer solution. The limitations which were imposed on the human computist in this case correspond to the limitations of an automatic computer. It is not a "brain", as has been suggested in some newspaper articles on the subject; it can only perform routine numerical operations as specified by a program which has been prepared for the particular problem under consideration.

In order that this simple calculation might be carried out by automatic computer, the engineer first would have to write out a calculation program in a form which the machine could understand (usually by a code punched on cards or paper tape) and feed it into the machine. Then he would have to prepare the basic data and input it similarly. Finally, pressing the start button would cause the machine to go through the same operations done by the computist above (but at a very much greater speed, of course). The machine program might be somewhat as shown in Fig. 3, clearly the exact equivalent of the computist's program.

Automatic Computer Program

a. Read "d" into storage space 1
b. Read "O" into storage space 2
c. In Table storage, locate value of "t" corresponding with number in storage space 2 and transfer to storage space 3
d. Transfer numbers from storage spaces 1 and 3 to arithmetic unit, multiply, and transfer product to storage space 4
e. Print out number contained in storage space 4

Fig. 3 Program for Automatic Computer Solution

This example demonstrates the functions of the essential components of an automatic digital computer, namely: (1) facilities for reading in instructions and data, (2) storage facilities to keep data and instructions available for use, (3) an arithmetic unit to carry out the actual numerical operations (usually limited to add, subtract, multiply and divide) and (4) a print-out device to present results in a usable form (usually typed on a sheet of paper). The additional operation involved in this problem of looking up the tangent of the angle, might be done by referring to a large scale storage unit in which a complete set of trigonometric functions is stored for reference, or it might be handled by a special sub-program which calculates the value of the tangent to the required accuracy by means of a series expansion.

The program required for this example calculation is too simple to indicate the complexity of the programming problem in general. At the present time, the preparation of the program is the principal restriction to widespread application of computing machines to structural analysis. To prepare and check out a program for a really complex problem may take weeks or even months of concentrated effort. For such cases, it is clear that there will be little advantage to be gained from a machine solution unless the same program can be used time after time with different data. However, as libraries

of such standard programs are built up, more and more structural work can be done by machines with no additional programming required. One such standard structural analysis program will be described in the last part of this paper.

Computer Methods of Structural Analysis

General Consideration

Before entering into a discussion of methods of structural analysis suitable for use with automatic computers, it will be desirable to review and outline the general nature of the structural problem. For present purposes, a structure may be defined as an assemblage of simple structural elements interconnected at a finite number of joints. The problem of structural analysis involves the evaluation of the forces in the elements and the displacements of the joints of the structure caused by application of a specified system of forces to the joints.

A correct solution of the structural problem must satisfy the following three requirements:

(1) Equilibrium: The system of internal and external forces acting on each joint of the structure must be in equilibrium.
(2) Compatibility: The deformations of the various elements of the structure must be so related that they will fit together under load.
(3) Force-deflection Relationships: The force and deformation in each element of the structure must be related according to a specific, predeterminable law, dependant upon the elastic properties and configuration of the element.

For linear structures, i.e., structures in which the forces and deflections of the elements are linearly related, the formulation of these conditions results in a set of linear simultaneous equations. The complete problem of structural analysis thus may be seen to consist essentially of the formulation and solution of these equations.

In considering the application of automatic computers to the solution of structural problems, the basic factor to be borne in mind is that the machines are capable of performing only very simple mathematical operations. Consequently, any approach to the analysis ultimately must be reduced to a sequence of simple routine steps. Otherwise, the only requirements imposed on the machine solution are the three conditions listed above, and any solution which satisfies these conditions must be considered satisfactory.

Matrix Formulation of the Load-Deformation Relationships

Probably the simplest approach to the systematizing of structural calculations for machine solution lies in the use of matrix algebra. For present purposes it will be necessary to understand only the matrix notation, and this will be described with reference to the force-deflection characteristics of the structural elements. (An excellent brief summary of simple matrix operations is presented in Bibliography Item No. 5.)

The force-deflection characteristics of a structural element may be expressed in either of two ways: the deflections may be expressed in terms of the loads or the loads in terms of the deflections. Considering the first

approach, by the principal of superposition the deflection in a given element at any point "a" may be expressed as a linear function of the loads applied at points a, b, ...n, thus:

$$\upsilon_a = f_{aa} S_a + f_{ab} S_b + \cdots + f_{an} S_n$$

in which: υ_a = deflection at point a
$S_a, S_b, \cdots$ = forces at points a, b, ...
$f_{aa}, f_{ab}, \cdots$ = flexibility influence coeffs. (1a)

Similarly, the deflections at the other points can be expressed:

$$\begin{aligned} \upsilon_b &= f_{ba} S_a + f_{bb} S_b + \cdots + f_{bn} S_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \upsilon_n &= f_{na} S_a + f_{nb} S_b + \cdots + f_{nn} S_n \end{aligned} \qquad (1b)$$

In matrix form this set of equations would be written:

$$\begin{Bmatrix} \upsilon_a \\ \upsilon_b \\ \cdot \\ \cdot \\ \upsilon_n \end{Bmatrix} = \begin{bmatrix} f_{aa} & f_{ab} & \cdot & \cdot & f_{an} \\ f_{ba} & f_{bb} & \cdot & \cdot & f_{bn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{na} & f_{nb} & \cdot & \cdot & f_{nn} \end{bmatrix} \begin{Bmatrix} S_a \\ S_b \\ \cdot \\ \cdot \\ S_n \end{Bmatrix} \qquad (2a)$$

or for general discussions of flexibility and deflections the matrix equation could be abbreviated to the following:

$$\underline{\upsilon} = \underline{f}\ \underline{S} \qquad (2b)$$

where the underlining indicates that the symbol represents a matrix.

Alternatively, the forces in the structure may be expressed in terms of the deflections at the load points, in entirely equivalent fashion. Thus the force acting at point "a" is a linear function of the deflections at all of the points:

$$S_a = k_{aa} \upsilon_a + k_{ab} \upsilon_b + \cdots + k_{an} \upsilon_n$$

in which: $k_{aa}, k_{ab}, \cdots$ = stiffness influence coeffs. with other notation as in Eq. 1a (3a)

Similarly, the forces at the other points may be expressed:

$$\begin{aligned} S_b &= k_{ba} \upsilon_a + k_{bb} \upsilon_b + \cdots + k_{bn} \upsilon_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ S_n &= k_{na} \upsilon_a + k_{nb} \upsilon_b + \cdots + k_{nn} \upsilon_n \end{aligned} \qquad (3b)$$

In matrix form, these equations would be written:

$$\begin{Bmatrix} S_a \\ S_b \\ \cdot \\ \cdot \\ S_n \end{Bmatrix} = \begin{bmatrix} k_{aa} & k_{ab} & \cdot & \cdot & k_{an} \\ k_{ba} & k_{bb} & \cdot & \cdot & k_{bn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ k_{na} & k_{nb} & \cdot & \cdot & k_{nn} \end{bmatrix} \begin{Bmatrix} \upsilon_a \\ \upsilon_b \\ \cdot \\ \cdot \\ \upsilon_n \end{Bmatrix} \tag{4a}$$

or in abbreviated form:

$$\underline{S} = \underline{k}\ \underline{\upsilon} \tag{4b}$$

The matrices of the flexibility and the stiffness influence coefficients $\underline{f}$ and $\underline{k}$ are called the flexibility and stiffness matrices of the element, respectively. For simple structural elements, these coefficients may be derived readily: for a flexural element, for example, the stiffness influence coefficients are the coefficients of the well-known slope-deflection equations.

The Two Fundamental Methods of Structural Analysis

Corresponding with the two methods of expressing the load-deformation characteristics of the elements, there are two basic approaches to the problem of structural analysis. In the more frequently used approach, commonly called the "force method", the (redundant) forces in the structural elements are taken as the unknown quantities, displacements in the members are expressed in terms of the (known and unknown) forces by means of the flexibility matrices, and expressions for the conditions of continuity (i.e., that all of the members fit together properly under load) provide a set of equations which is solved simultaneously for the unknown forces.

The second basic approach used in structural analysis, commonly called the "displacement method", is essentially the inverse of the first. The displacements of the joints of the structure are taken as the unknown quantities, the forces in the members are expressed in terms of these unknown displacements by means of the stiffness matrices, and the conditions of equilibrium (i.e., that the forces acting on each of the joints in the direction of each possible component of displacement must balance) provide the equations which are solved simultaneously for the unknown displacements. The slope-deflection method is a good example of this type of approach.

Both of these approaches may be formulated conveniently in matrix form suitable for treatment by automatic computing machines, as shown in the Appendix. The decision as to which approach will be most suitable for a given problem depends upon several factors. If the work is to be done manually, the solution of the set of simultaneous equations usually represents a large part of the total work to be done in any sizeable problem, and the choice of method generally is based upon the relative number of force unknowns (redundants) in the system as compared with the number of displacement unknowns. With an automatic computer solution, however, the time required to set up the basic matrices of the system may be a more important factor than the number of equations to be solved. For example, the computer at the University of California can solve 45 simultaneous equations for 45 unknowns in 3-1/2 minutes. Thus it clearly would make little difference whether the

solution of a given system involved, say, 6 or 15 simultaneous equations. Similarly, it is as easy for the machine to evaluate the flexibility matrices of the elements as it is to evaluate the stiffness matrices. Consequently, for machine solutions, the choice of approach frequently will be decided on the only other factor entering the analysis: the ease of setting up the static relationships involved in the force approach as compared with setting up the kinematic relationships involved in the displacement approach. In some cases, it will be found easier to express the member deformations in terms of the joint displacements (kinematics) than it is to express the member forces in terms of the joint loads (statics) suggesting that the displacement method is to be preferred, while in other cases the opposite situation will exist.

One other factor which apparently should be considered in the choice of approach is the maximum number of simultaneous equations which can be handled by the machine. Due to storage limitations any given machine can solve only a certain number of equations, and it would seem that this fact might dictate the choice of method to be used in some cases. However, by either the force or displacement approach it is possible to obtain the solution of large complex systems by first analyzing subsections of suitable size and then combining these partial solutions. Thus the size limitation actually is of secondary importance. (It may be of interest to note, in passing, that in the force approach, the process of analyzing systems by parts amounts to the use of a statically indeterminate primary structure, the first stage of the solution being the analysis of the primary structure itself.)

Examples of Computer Analyses

Rigid Frame Analysis

Matrix formulations of both the force and displacement methods of structural analysis have been used extensively in automatic computer solutions of structural problems during the past few years. Much of this work has been done in the aircraft industry where the emphasis has been upon the analysis of stressed skin structures. The present discussion will be limited, however, to machine analyses of structures composed of beam and column elements which are more typical of civil engineering practice.

Probably the most extensive literature on computer analyses of such structures has been published by Dr. R. K. Livesley, now of Cambridge University in England, and the material presented here is based primarily on these publications. In his work, Dr. Livesley set out to develop a standard program which could be used for the analysis of any planar structure built up of members having uniform properties throughout their length (e.g. standard structural sections).

The program is based upon the displacement approach described above, and is worked out in four stages. The first stage is concerned with establishment of the stiffness matrices of the individual members. The data fed into the computer are the section properties (modulus of elasticity E, moment of inertia I, and area A) and the geometric quantities (length L and angle of the member from the horizontal α) for each member. The machine then proceeds to calculate the elements of the stiffness matrix for each member, and to store them internally. In the second stage of the analysis, the member stiffnesses are combined to give the joint stiffnesses, the data required for this operation being merely the designation of the members meeting at each joint.

At this time, the external loads acting on the joints are also incorporated into the expressions, the result being a set of simultaneous equations involving the unknown joint displacements. The third stage of the program is simply the solution of these simultaneous equations to obtain the joint displacements. This as a routine operation having many uses in fields other than structural analysis, of course, and generally is included among the standard library routines maintained with most computers. The final stage of the calculation is the evaluation of the member forces from the joint displacements. This is a simple matter since the stiffness matrices, which already have been evaluated, express the member forces in terms of these displacements.

Due to the fact that the stiffness matrices of the elements take account of axial as well as bending stiffness, this standard program is suitable for the analysis of truss structures as well as rigid frames. In the analysis of a truss with rigid joints, the program will automatically obtain the bending moments in the members associated with secondary stress. Similarly, in the analysis of a rigid frame, the effect of axial deformations in the members will be accounted for automatically. To carry out the analysis of a pin-jointed structure, the members are assumed to have zero bending stiffness, while to carry out a solution neglecting the effect of axial deformations, a very large value is used for the cross-sectional area of the members.

An example of a problem solved by Dr. Livesley with this program is shown in Fig. 4, a rigid frame of the type encountered in power station building structures. This frame has 11 joints (excluding the rigidly fixed column footings) and consequently has 33 degrees of freedom. It was analyzed for five different conditions of loading, including vertical, wind, and various crane loadings.

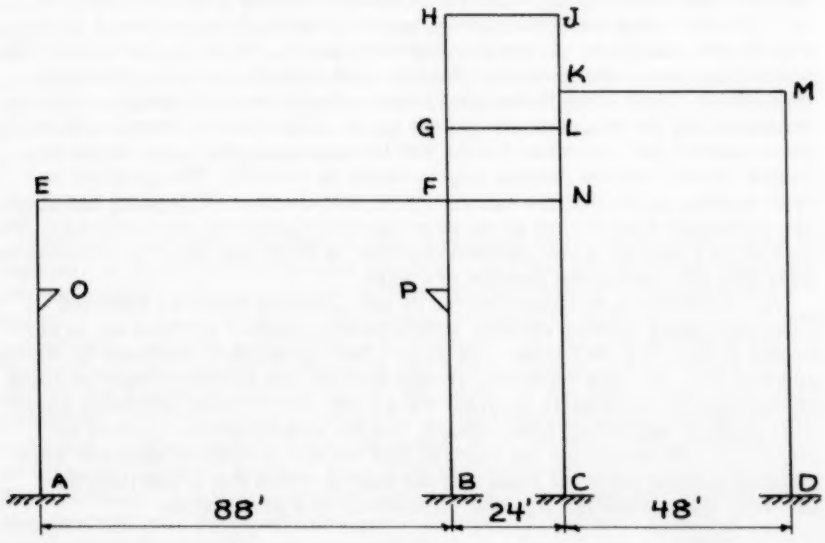

Fig. 4 Rigid Frame for Power-House Structure

The manual labor required to prepare this problem for solution on the machine consisted of:

a) Tabulating the length, area, and moment of inertia of each member.
b) Determining the equivalent joint loads induced by each condition of loading.
c) Punching and checking the data tapes from the coded information.

Time required for the first three operations was about 14 man-hours, including checking, while the preparation of the data tapes took an additional 6 man hours. The computer time required for the solution of each loading case was about 15 minutes, making a total for the problem of about 1-1/4 hours machine time. Results obtained from the analysis for each loading condition included the displacements of all the joints (rotations as well as linear movements) and the moment, shear and axial force in each member at each joint. With regard to the total of 20 man-hours required for preparation of data for this problem, Dr. Livesley pointed out that the work was done by relatively inexperienced personnel, and could have been done considerably faster by experienced operators. It might also be mentioned that the Manchester University Computer used in this work is a relatively slow machine, thus the machine time should be much less if the computation were done on one of the faster machines available in this country.

Truss Stability

With slight modifications this program can be used also to analyze structures in which axial forces in the members have a significant effect on their bending stiffness, i.e. cases where "in-plane" buckling is a problem. To consider such cases, it is necessary only to include the axial force effect in the evaluation of the bending stiffness of each member. Because the true axial forces in the members are not known initially, it is necessary to approach the solution by successive approximations. First, neglecting the effect of axial force on the member stiffnesses, the axial forces are calculated throughout. Then using these axial forces, the member stiffnesses are re-evaluated and the analysis carried out again. After two or three such cycles, the computed axial member forces will be essentially the same as the assumed values, and the results may be taken as correct. The program has been worked out so that the calculation of new member stiffnesses based on the calculated axial forces of the previous cycle proceeds automatically. Thus, all that is required of the machine operator is to decide when the machine has gone through a sufficient number of cycles.

An example of a structure solved by Dr. Livesley with this extended program, taking account of axial effects on the member stiffnesses, is presented in Fig. 5, a roof truss with rigid joints (previously analyzed by Winter, and others). For this relatively simple system, the machine required about 12 minutes for a complete analysis for a given set of loads, including the initial analysis neglecting axial effects, and the subsequent successive adjustments. To determine the critical load for the structure, analyses were made at several different loads and the load at which one of the joint deflections approached infinity was determined by extrapolation.

Optimum Design

One of the great advantages of the machine solution of structural problems

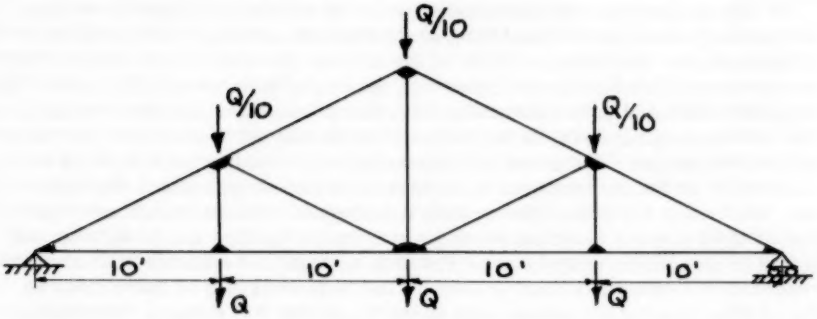

Fig. 5 Rigid Jointed Roof Truss

is that it provides a practical means to obtain optimum designs of statically indeterminate systems. Once the problem has been set up, it is a relatively simple matter to change the dimensions of the members at critical regions in the structure and re-analyze successively until the most efficient structural arrangement has been achieved. Livesley also has modified his program slightly for this purpose to simplify the selection of various member sizes. Rather than feed in the member sizes as initial data, with this modified program he arranged to store the properties of 96 standard sections in the storage unit of the machine. Then the operator is able to select the desired member for each part of the structure simply by pressing a button on the machine corresponding to that storage position. The final stage of the analysis program was also modified so that instead of printing out axial forces and moments in the member, the resultant member stress was given. Thus the operator can select a set of member sizes by pressing the appropriate storage buttons, let the machine run through the calculation, observe the resulting stresses, select new member sizes as indicated and repeat the process until a satisfactory design is achieved.

Structural Dynamics

Automatic computing machines also have found wide usage in predicting the response of structures to dynamic loadings. In many respects, the machines provide greater benefits in such applications than they do in ordinary static analyses because of the inherently greater complexity of the dynamic problem.

The basic dynamic properties of a structure, i.e. its characteristic vibration mode shapes and frequencies, may be evaluated conveniently by extending the matrix procedures described above for the static analysis. These properties are very useful in determining the response of structures to harmonically varying applied loads. However, when the structure is subjected to an irregular dynamic loading, such as might be developed by earthquake or blast pressure, a different approach to the analysis, known as the step-by-step method, may prove advantageous. In fact, for cases in which the structure is strained beyond the yield point, the step-by-step method may well provide the only rational solution.

In this method, the successive displacements of the structure occurring during very short intervals of time are computed, starting with specified initial conditions and taking account of the applied load and structural resistance which is effective during each interval. As might be expected, this method is very laborious and time consuming when the calculations are done manually, but automatic computers, fortunately, are well adapted to such routine calculation procedures. The power of the computer in handling such work is well illustrated by its performance in a research problem with which the writer was associated recently. The system under consideration in this case consisted of 63 masses interconnected by non-linear springs and in addition subjected to non-linear drag forces. For this system, the machine evaluated the response to a specified loading through 100 time intervals in three minutes, the greater part of which time was spent in printing out results. The magnitude of this performance is indicated by the fact that a first-rate computist spent over a month carrying out about a tenth of these same calculations on a desk calculator for check purposes. It must also be realized, of course, that many man-weeks of labor went into the preparation of the machine program used in these calculations. Development of the program was economically feasible only because it was to be used in evaluating results for approximately one hundred problems of this type.

Concluding Remarks

It has been the purpose of this presentation to indicate briefly how an automatic computer works, and how it may be used in the solution of structural problems. Before closing, it should be mentioned that, although the matrix procedures discussed here offer many advantages in the programming of structural problems for machine solution, they are not the only approaches which may be used. Moreover, the fact that Dr. Livesley's work has been used for discussion in this paper should not be taken to imply that no significant developments in automatic computer solutions of civil engineering structures have been made in this country. A considerable amount of work of this type has been and is being done here, but unfortunately the publication of results has generally been lagging far behind the status of the work.

In conclusion, some of the advantages to be gained by a structural engineering office through the use of automatic computers will be reviewed. There are three principal benefits to be derived from the use of automatic computers. First, and probably of greatest immediate importance from a business standpoint, is the matter of economy. In general, calculations done by a machine cost considerably less than the same work done by hand. The American Bridge Company, which probably has more experience than any other organization in this country with the machine solution of civil engineering structural problems, has found that calculations done by their machine cost only one quarter as much as the corresponding hand calculations. In other words, in spite of the relatively high cost of the machine (including rental, programming, etc.) its high speed enables it to save 75 per cent of the usual cost of structural analysis work.

A second benefit inherent in the use of computers is the increase which they make possible in the productivity of the engineers working with them. During the present (and expected future) shortage of engineering personnel, this factor is of great importance. As was mentioned earlier, a machine can

do in minutes what it might take an engineer weeks or even months to do. Moreover, the reliability of the machine calculations further increases the productivity of the engineer by relieving him of the necessity of checking the calculations. For most problems, checks can be built right into the calculation program so that the results may be considered as reliable as the data fed into the machine. Thus, the use of a computer relieves the engineer of much routine computational work and leaves him free to concentrate on the real engineering aspects of the job; in effect, the engineer's position is upgraded at the same time that his productivity is amplified.

The third future benefit which may be expected to result from the use of automatic computers is the ability to treat problems which otherwise could not be considered or could be analyzed only very roughly. In this connection, it must be kept in mind that the application of electronic computers in the solution of structural problems still is in its infancy. Large, high-capacity machines have been available for such work for only a few years, and continuous improvements are being made in machine methods and techniques. One area in which significant advances should take place in the immediate future is the optimization of structures which was discussed briefly above. Other developments can be expected in the types of methods which are available for machine analyses. It is likely that new methods which take full advantage of the capabilities of automatic computers will be devised, making possible the exact analysis of structures which now might only be designed on the basis of model studies or crude approximations. In this regard, it may be noted that the analytical procedures which have been developed in the past few decades for efficient manual solutions cannot be expected, generally, to prove equally efficient in a machine solution. These automatic digital computers make possible a completely new approach to calculation, and the development of methods of effectively utilizing their capabilities in structural analysis presents a tremendously stimulating challenge to forward looking engineers.

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Schureman, L. R., "Increasing Productivity of Engineers through the Use of Electronic Computers"
Zahler, C. W., "The Electronic Computer in Bridge Design"

(b) Presented at the Western Regional Conference on Increasing Highway Engineering Productivity, Los Angeles, March 1957.

Chafetz, E. M., "Current Experiences with Electronic Computations"
Chang, J. C. L., "The Use of Electronic Computers in the Design of Two Tied Arch Bridges"
Marmelstein, C. A., "Progress Report on the Use of the Digital Computer in Bridge Work in Georgia"
Schureman, L. R., "The Use of the Electronic Computer in Continuous Beam Bridge Design"

Appendix

Matrix Formulation of Structural Theory

Many authors have contributed to the matrix treatment of structural problems during the past two decades, as is evidenced by list of papers contained in the Bibliography. Probably the most thorough and comprehensive treatment of the subject was presented by J. H. Argyris in his series of articles in Aircraft Engineering (Reference No. 1). The formulations of both methods of structural analysis presented here are based on Argyris' work, and make use of his notation.

The Force Method

In this approach, the first phase of the analysis is the establishment of a flexibility matrix $\underline{f}$ and two force transformation matrices b_0 and b_1. The flexibility matrix is a diagonal matrix of the submatrices $\underline{f_1}, \underline{f_2} \ldots \underline{f_m}$,

$$\underline{f} = \begin{bmatrix} \underline{f_1} & 0 & \cdot & \cdot & 0 \\ 0 & \underline{f_2} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \underline{f_m} \end{bmatrix} \qquad \text{(A-1)}$$

thus where the submatrices $\underline{f_1}$, etc., are the flexibility matrices of the individual elements (beams, columns, etc.) of which the structure is constructed.

To establish the force transformation matrices, it is first necessary to make the system statically determinate by selecting a set of redundant forces (one for each degree of static indeterminacy). The force transformation matrix $\underline{b_0}$ then represents the statically determinate equilibrium relationships between the external loads applied to the structure and the forces developed in each element, the redundants being assigned some arbitrary values (usually zero). Similarly, the matrix $\underline{b_1}$ represents the statically determinate equilibrium relationships between the redundant forces and the element forces.

When the flexibility of the elements and the force transformation matrices have been established, the analysis of the structure is carried out by a series of matrix operations as follows:

1) Evaluate the displacements corresponding to the redundants

$$\underline{D} = \underline{b_1'}\ \underline{f}\ \underline{b_1} \qquad \text{(A-2)}$$

2) Compute the values of the redundants due to application of unit external loads

$$\underline{X} = -\underline{D^{-1}}\ \underline{b_1'}\ \underline{f}\ \underline{b_0} \qquad \text{(A-3)}$$

3) Compute the stress matrix, i.e. the total forces in all the elements of the structure due to unit external loads

$$\underline{b} = \underline{b_0} + \underline{b_1}\ \underline{X} \qquad \text{(A-4)}$$

4) Compute the flexibility matrix of the structure, i.e. the displacements corresponding to the external loads due to application of unit external loads

$$\underline{F} = \underline{b_o'}\ \underline{f}\ \underline{b} \tag{A-5}$$

5) Check the structural continuity at the redundants when unit external loads are applied.

$$\underline{b_1'}\ \underline{f}\ \underline{b} \stackrel{?}{=} \underline{0} \tag{A-6}$$

After the stress and flexibility matrices have been obtained from Eqs. (A-4) and (A-5), the stresses $\underline{S}$ and deflections $\underline{r}$ due to any system of applied loads $\underline{R}$ may be calculated from the following equations:

$$\underline{S} = \underline{b}\ \underline{R} \tag{A-7}$$

$$\underline{r} = \underline{F}\ \underline{R} \tag{A-8}$$

The Displacement Method

As was mentioned in the body of the paper above, the displacement method is essentially the inverse of the force method. Matrix formulations of the two methods can be set up in completely parallel form, with stiffness and displacement in the displacement approach being equivalent respectively to flexibility and force in the force approach. On this basis, however, the inverse of the force method outlined above would involve obtaining a solution for specified external displacements of the structure, rather than specified external forces as is usually the case. Thus, the solution of a system for specified external forces by the displacement method is not exactly parallel with the force method solution of the same problem.

The first phase of the analysis, however, is quite equivalent. It involves establishing a stiffness matrix $\underline{k}$ and a displacement transformation matrix a_o. The stiffness matrix is the diagonal matrix of the submatrices $\underline{k_1}$, $\underline{k_2}$, ... $\underline{k_m}$, thus:

$$\underline{k} = \begin{bmatrix} \underline{k_1} & 0 & \cdot & \cdot & 0 \\ 0 & \underline{k_2} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \underline{k_m} \end{bmatrix} \tag{B-1}$$

where $\underline{k_1}$, etc. are the stiffness matrices of the individual elements of which the structure is constructed. To establish the displacement transformation matrix, it is necessary first to make the system kinematically determinate by fixing all of the joints. The matrix a_o then represents the kinematic relationships between the joint displacements and the member deformations, only one component of joint displacement being considered at a time.

When the stiffness and displacement transformation matrices have been established, the displacement analysis proceeds as follows:

1) Compute the stiffness matrix of the complete structure:

$$\underline{K} = \underline{a_o'}\ \underline{k}\ \underline{a_o} \tag{B-2}$$

2) By inverting the stiffness matrix, obtain the flexibility matrix of the complete structure:

$$\underline{F} = \underline{K^{-1}} \tag{B-3}$$

3) Calculate the stress matrix of the complete structure:

$$\underline{b} = \underline{k}\ \underline{a_o}\ \underline{K^{-1}} \tag{B-4}$$

This represents the complete analysis by the displacement method. The deflections and stresses in the structure caused by any set of externally applied forces $\underline{R}$ can be obtained directly from the flexibility and stress matrices $\underline{F}$ and $\underline{b}$ as was indicated in the discussion of the force method, Eqs. (A-7) and (A-8).

From this outline, it is apparent that the displacement method is simpler in principle than is the force method. In practice, however, the inversion of the matrix $\underline{K}$ (essentially the solution of a set of simultaneous equations) frequently is a very extensive operation because of the large number of joint displacement components which must be considered. (Usually there are many more joint displacements than statically indeterminate forces in the system.) For this reason, the displacement method may not be as advantageous as it appears.

In some cases, the stiffness matrix $\underline{K}$ may be condensed before it is inverted, and in this way the magnitude of the computation may be considerably reduced. This condensation is possible if there are some joint displacement components for which no corresponding forces are to be applied (i.e. certain elements of the external force matrix $\underline{R}$ are specified to be zero). In this case, the basic relationship between joint forces and displacements

$$\underline{R} = \underline{K}\ \underline{r} \tag{B-5}$$

may be written in the partitioned form:

$$\begin{Bmatrix} \underline{R_o} \\ \underline{0} \end{Bmatrix} = \begin{bmatrix} \underline{K_{oo}} & \underline{K_{o1}} \\ \underline{K_{1o}} & \underline{K_{11}} \end{bmatrix} \begin{Bmatrix} \underline{r_o} \\ \underline{r_1} \end{Bmatrix} \tag{B-5a}$$

The displacements $\underline{r_1}$ may now be eliminated from this relationship, resulting in the equation:

$$\underline{R_o} = \left[\underline{K_{oo}} - \underline{K_{o1}'}\ \underline{K_{11}^{-1}}\ \underline{K_{1o}}\right] \underline{r_o} \tag{B-5b}$$

from which it is seen that the desired condensed stiffness matrix $\underline{\tilde{K}}$ is given by:

$$\underline{\bar{K}} = \left[\underline{K_{00}} - \underline{K'_{01}}\ \underline{K_{11}^{-1}}\ \underline{K_{10}}\right] \qquad \text{(B-6)}$$

This condensation process may be considered as the solution of a kinematically indeterminate structure, because releasing the joint components which are not required in the flexibility matrix makes the structure kinematically indeterminate. Such an analysis can be shown to parallel exactly the solution of a statically indeterminate structure by the force method.

Paper 1637 May, 1958 ST 3

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SPECIFICATION FOR PRESERVATIVE TREATMENT OF TIMBER[a]

Progress Report of a Sub-Committee of the Committee
on Timber Structures of the Structural Division
(Proc. Paper 1637)

SYNOPSIS

The treatment of structural timber with preservatives has long been practiced by engineers and builders. Recent progress in materials and methods of treatment of timber has been documented in many publications. Not so readily available, however, are criteria for application of the various types of preservative. The purpose of this paper is to provide recommendations for usage in addition to brief descriptions of the various preservatives currently in common usage.

INTRODUCTION

Decay, insect attack and fire are natural enemies of wood. Sound engineering judgment calls for pressure preservative treatment of wood products such as piling, poles, lumber, ties, timbers and plywood which may be exposed to any of these hazards. The only satisfactory application of preservatives is by pressure processes, thus a knowledge of the recognized preservatives and the methods of treating are essential.

Treatment Methods

The method of pressure treatment used should be that which will provide the best product for the purpose intended with the preservative specified. This is done by either the empty cell or full cell process.

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a. This paper will form the basis for a chapter in a proposed revision to ASCE Manual of Engineering Practice No. 17 entitled "Timber Piles and Construction Timbers."

Empty Cell Process

The empty cell process starts with the charge either at atmospheric or under pressure. The cylinder is then filled with preservative imprisoning air within the wood. Combinations of additional pressure and temperature force preservative deep into the wood. Accurate gauging measures the amount of preservative pressed into the charge. Pressure is then released, the preservative drained from the cylinder and a final vacuum removes the remaining surplus preservative from the wood. The initially imprisoned air helps expel the excess treating solution, resulting in a drier product.

This process should be specified for oil treatments except where material is subject to marine borers or where the retention required is greater than can be obtained by this process. It results in a maximum penetration for a given amount of preservative and should be specified for railroad ties, poles, posts, bridge piles and timbers.

Full Cell Process

The full cell process starts with a vacuum to expel all air from the wood. The treating cylinder is then filled with preservative and the pressure is raised and maintained until the specified retention is assured or the wood treated to refusal. Pressure is then released, preservative drained from the cylinder and a final vacuum may be created to remove excess preservative. This process should be specified for oil treatments to be used in marine waters or conditions of severe exposure such as wharf and pier piles and timbers and bracing within the tide zone. This method should be used wherever the maximum amount of preservative is desired to be retained in the wood. Water borne preservatives are generally impregnated by the full cell process.

Preparation for Treatment

Wood that is to be pressure tested should be sound and free from all defects that would render it unfit for the purpose intended. Practically all species of wood are amenable to pressure treatment although some may be slightly more refractory than others. The sapwood of all species is most susceptible to attack, but is more easily penetrated by the preservative than the more refractory heartwood. Sapwood and heartwood have equal strength so the sapwood content should not be limited for material which is to be pressure treated.

Round materials should be peeled to remove all outer and inner bark before treating. This helps avoid decay or insect attack while air seasoning assures more uniform seasoning and results in a more satisfactory treated product.

The materials in any one cylinder charge should be as uniform as possible as to moisture content, species, form and dimension.

Conditioning

Conditioning before treatment can be accomplished either by air seasoning, kiln drying (usually 2" or less in thickness) or, steaming and heating in the preservative. If artificial seasoning is necessary, the method used is dependent upon the type of plant equipment available.

Boiling under vacuum: is a method of conditioning unseasoned material in preparation for pressure treating. It consists of heating the timber in the preservative oil while under a vacuum. Moisture removed as steam during this boiling is condensed and measured to determine when the material is ready for pressure treatment in the regular manner.

Steam and vacuum: process is another method of conditioning unseasoned material. As the name implies, alternate cycles of steaming or vacuum are used to reduce the moisture content of the charge.

Vapor drying: consists of subjecting the green wood to vapors produced by boiling organic chemicals and removing the resultant mixed water and chemical vapors from the cylinder.

Fabrication

All machining, consisting of framing, boring, adzing, chamfering, gaining, surfacing or trimming should be accomplished prior to treatment. This eliminates any necessity for damage to the treated zone by subsequent field cutting.

To protect intricate fabrication, incising should be accomplished prior to this operation.

Incising

To insure uniform penetration, all sawn material 3" or thicker should be incised on all faces. Thinner material may be incised on the wide faces only.

This operation separates the outer fibers by means of knife-like teeth. The effect on the structural strength is negligible and it facilitates uniform penetration of the preservative, minimizes checking, and aids in relieving internal stress occasioned by changing moisture content.

Field Practices

Pressure treated timber should be handled carefully so the protective zone provided by the pressure preservative treatment is not pierced during handling.

Any untreated wood which is exposed by cutting after pressure treatment, should be given two coats of hot preservative, allowing sufficient time between coats for reasonable absorption.

Pile cutoffs should be field treated, preferably by creating a reservoir for hot creosote 1" or more in depth over the untreated core of the cut-off ends. Light gauge metal rings, mud dikes or a series of shallow bored holes will help form a reservoir to retain the creosote. Good penetration can be obtained as the end grain of wood is quite absorbent. The pile cutoff should then be protected by a built-up roofing cap.

Wood Treating Chemicals

Wood preservatives are chemicals that are stable, extremely toxic to agencies that destroy the wood or its fibers, have a high degree of permanence, possess qualities permitting good penetration and are harmless to wood as well as any metal with which they might be in contact. The treatments are of four classifications.

(1) Creosote or creosote solutions:

Creosote
Creosote Coal-Tar Solution
Creosote-Petroleum Solution

These are of low volatility and are practically insoluble in water so are most suitable to severe exposures. Creosote and creosote coal-tar solutions are the only preservatives presently recommended for use in marine structures subject to marine borer attacks.

(2) Oil-Borne chemicals:

Pentachlorophenol
Copper - Naphthenate

These are organic salts dissolved in a petroleum carrier oil. They are suitable for outdoor exposure or use in contact with the ground.

(3) Inorganic salts:

Chromated Zinc Chloride
Copperized Chromated Zinc Chloride
Tanalith (Wolman Salts)
Acid Copper Chromated (Celcure)
Ammoniacal Copper Arsenite (Chemonite)
Chromated Zinc Arsenate (Boliden Salt)
Chromated Copper Arsenate (Greensalt or Erdalith)

These are salts dissolved in water or ammonia that evaporates after treatment leaving the chemicals as a precipitate in the wood. The treating solution strengths are varied to achieve the net dry salt retentions required.

Generally they are less resistant to leaching than creosote and oil borne preservatives but good service life records are being established in various types of installations including ground contact.

Where a clean, odorless, paintable surface is required, these preservatives may be specified.

(4) Fire retardant chemicals:

Proxtexol
Minalith
Pyresote
Chromated Zine Chloride (FR)

These chemicals enable wood to resist its own combustion by fire. They are intended for interior or protective use. If exposed to leaching conditions, they should be protected, preferably with a non-volatile sealer.

Specifications

Each type of pressure treatment and classification of preservatives has certain distinct advantages. Specifications for the type of chemical and its application should be determined by the required end use of material.

A recommended specification should state the preservative or type of preservative permitted and the minimum retention desired. The pressure treatment should be in accordance with the standards published by the American Wood Preservers' Association in their manual of recommended practices.

Actual service records have established minimum net retentions that should be specified for the various preservatives covering each condition of use. Table 1 illustrates recommended minimum retentions.

Respectfully submitted,

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Sub-Committee of the Committee on Timber Structures of the Structural Division

TABLE 1

Recommended Minimum Net Retentions
As Submitted by W. Bond

Creosote and Creosote Solutions

Form of product and service condition	Minimum net retention of preservative recommended in lbs/cu. ft.		
	Coal-tar creosote	Creosote-coal-tar solutions	Creosote-petroleum solution
Ties			
Crossties, switch ties, bridge ties	8	8	9
Lumber, structural timber and plywood			
For use in coastal waters:			
Pacific Coast Douglas-fir	14	14	
Southern yellow pine	20	20	
For use in fresh water, in contact with the ground, or for important structural members not in contact with the ground or water	10	10	12
For other use not in contact with the ground nor in water	6	6	7
Piles			
For use in coastal waters:			
Pacific Coast Douglas-fir	14	14	
Southern yellow pine	20	20	
For land or fresh-water use:			
Douglas-fir; oak, red and white; pine, jack, lodge-pole, ponderosa, red and southern yellow	12	12	14
Poles	8		
Posts	6	6	7
Crossarms	6		

Oil Borne Chemicals

	5% Pentachlorophenol solution in petroleum	Copper Naphthenate solution in petroleum (0.5% Copper)
Lumber, structural timber and plywood		
For use in contact with the ground	10	10
For use not in contact with the ground nor in water	6	6
Poles	8	8
Posts	6	6
Crossarms	6	–

TABLE 1 (Continued)

Inorganic Salt

	Boliden Salts (Cromated zinc arsenite)	Celcure (acid-cupric-chromate)	Chemonite (ammoniacal copper arsenite)	Chromated-zinc chloride	Wolman salts Tanalith	Copperized chromated zinc chloride
Lumber, structural timber and plywood						
For use under moderate leaching conditions	1.00	0.75	0.45	1.15	0.55	1.15
For use not in contact with the ground nor in water	.50	.50	.30	.75	.35	.75
Posts	1.00	.75	.45	1.15	.55	1.15

Fire Retardant Chemicals

	Minalith, Pyresote,* Protexol or Chromated Zinc Chloride* (FR)	
	Moderate fire retardant	High fire retardant
Lumber, structural timber and plywood		
Net thickness		
Plywood	3.0	6.0
Not over 2" thick	2.5	5.0
Over 2" but not over 4" thick	2.0	4.0
Over 4" but not over 6" thick	1.75	3.5
Over 6" but not over 8" thick	1.5	3.0
Over 8" but not over 12" thick	1.25	2.5
Over 12" but not over 16"	0.75	1.5
Over 16" thick	0.5	1.0

These chemicals are intended for interior or protected use. For exposed use materials should be protected with a sealer.

* Have preservative qualities.

Paper 1638 May, 1958 ST 3

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LATERAL LOAD ANALYSIS OF TWO-COLUMN BENTS

John E. Goldberg,[1] M. ASCE
(Proc. Paper 1638)

SYNOPSIS

The derivation of a simple set of simultaneous equations involving only the joint rotations in a symmetrical two-column bent under lateral loads is reviewed. These equations ordinarily are well-conditioned for a rapid iterative solution, even in cases of rather extreme ratios of column stiffness to girder stiffness. The form of the equations is such that an algebraic solution may also be easily obtained. The method of solution is directly applicable to certain Vierendeel girders. Both iterative and formal solutions are demonstrated.

INTRODUCTION

Several practicable and valid methods are available for the analytical determination of stresses due to wind and other lateral forces in the general case of tall building bents. Among these are several variations of the moment distribution procedure(1) and the author's iterative solution(2) of the slope deflection equations. The effectiveness and simplicity(3,4) of these methods is such that there is little justification for patently approximate methods when the engineer is confronted by general problems of this type.

In the case of a symmetrical two-column bent, still more effective methods of solution may be developed. Moderately tall buildings with two-column wind bents, such as the new Inland Steel Company Building in Chicago, are already in existence and others are in the design stage. In view of current and possible future interest in symmetrical two-column bents, it is appropriate to examine simplified methods for the analysis of this type of structure under lateral load.

The basic equations upon which these methods may be based

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were published[5,6] previously, and were derived several years earlier in connection with the work of the ASCE Subcommittee on Wind Bracing. It is probable that similar equations have been presented by others. When the equations were first derived it was found that they provided the basis of a simple iterative solution which converged with remarkable rapidity. Accurate solutions were thus readily obtainable even in cases where the columns were eight to ten times as stiff as the girders and for which the usual shear distribution method is not too effective.

This method of solution is practically self-checking. It is only necessary to note whether the calculated moments at each joint sum up to zero. The lateral displacements, if desired, are easily computed. It may be noted that the method of solution is directly applicable to the related form of Vierendeel girder.

Notation

The following symbols are used:

A_n, a derived constant for the n-th story;

E, modulus of elasticity;

h_n, height of n-th story;

I, moment of inertia of cross section;

K, relative stiffness; K_{AB}, relative stiffness of the member AB; K^C_{nm}, relative stiffness of one column of the n-th story; K^G_n, relative stiffness of the girder at the top of the n-th story;

k, factor of proportionality between relative stiffness and true stiffness (2EI/L);

L, length of a member;

M, moment; M_{AB}, moment at the end A of the member AB; M^C_{nm}, moment at the top of one column of the n-th story; M^G_n, moment at the end of the girder located at the top of the n-th story; M_n, external shear (S_n) at the n-th story times the story height (h_n);

m, n, o, subscripts denoting three successive stories of the bent in ascending order;

R, slope of line joining end points of a member, in the anomalous units used in the solution; corresponds to δ/L which is in radian units;

S_n, external shear at the n-th story of the bent;

Δ_n, lateral displacement at top of n-th story;

δ, relative lateral displacement at the ends of a member;

ϕ, angular rotation of a joint, in radians;

θ, angular rotation of a joint, in anomalous units;

θ', initial estimated value of θ.

Derivation

If the ends of a prismatic member (AB) are rotated through angles θ_A and θ_B respectively and are simultaneously given a relative lateral displacement δ, the moment exerted by the end A upon the adjacent joint is given by the slope deflection equation(5)

$$M_{AB} = -\frac{2EI}{L}(2\phi_A + \phi_B - 3\frac{\delta}{L}) \tag{1}$$

Ordinarily, because of the magnitudes involved, it is not convenient to use the stiffness, 2EI/L, in in-lb units or to compute the rotations in radians. Instead, a conveniently small relative stiffness K may be used and the rotations may be expressed in somewhat anomalous units such that the product of relative stiffness times rotation automatically is in units which are consistent with the applied moments, for example, kip-ft. With these modifications, Eq. 1 becomes

$$M_{AB} = -K_{AB}(2\theta_A + \theta_B - 3R_{AB}) \tag{2}$$

In dealing with wind stresses in tall building bents, it is usual to assume that the loads are applied as concentrations at the levels of the girders. With this assumption, the rotations of the two joints at any given level of a symmetrical two-column bent become equal.

Consider either column of the n-th story. The rotations at the bottom and at the top of that column are θ_m and θ_n, respectively; and the rotation of the line joining the ends of the column is R_n. By Eq. 2 the moments at the bottom and at the top of the column are

$$M^c_{mn} = -K^c_n(2\theta_m + \theta_n - 3R_n) \tag{3}$$

$$M^c_{nm} = -K^c_n(2\theta_n + \theta_m - 3R_n) \tag{4}$$

Similarly, the moment at the bottom of the column immediately above the n-th girder is

$$M^c_{no} = -K^c_o(2\theta_n + \theta_o - 3R_o) \tag{5}$$

The shear in either column of the n-th is the sum of the moments at the ends of the column divided by the story height, and must be equal to one-half of the external shear at the n-th story. Thus,

$$M^c_{mn} + M^c_{nm} = \frac{S_n h_n}{2} \tag{6}$$

$$\text{or } M^c_{mn} + M^c_{nm} = \frac{M_n}{2} \tag{7}$$

By use of Eqs. (3) and (4), this may be written as $-K_n^c(3\theta_n + 3\theta_m - 6R_n) = \frac{M_n}{2}$ which leads to

$$R_n = \frac{M_n}{12K_n^c} + \frac{1}{2}(\theta_n + \theta_m) \tag{8}$$

Substitution of Eq. (8) in Eqs. (4) and (5) yields

$$M_{nm}^c = \frac{M_n}{4} - \frac{K_n^c}{2}(\theta_n - \theta_m) \tag{9}$$

$$M_{no}^c = \frac{M_o}{4} - \frac{K_o^c}{2}(\theta_n - \theta_o) \tag{10}$$

Considering now the girder at the top of the n-th story, the moment at either end of this girder is

$$M_n^G = -K_n^G(2\theta_n + \theta_n)$$

$$\text{or } M_n^G = -K_n^G(3\theta_n) \tag{11}$$

Equilibrium requires that the sum of the moments acting on each joint be zero. Thus

$$M_n^G + M_{nm}^c + M_{no}^c = 0 \tag{12}$$

Substitution of Eqs. (9), (10) and (11) in Eq. (12) yields

$$(6K_n^G + K_n^c + K_o^c)\theta_n = \frac{M_n + M_o}{2} + K_n^c\theta_m + K_o^c\theta_o \tag{13}$$

$$\text{or } \theta_n = \frac{M_n + M_o}{2A_n} + \frac{K_n^c}{A_n}\theta_m + \frac{K_o^c}{A_n}\theta_o \tag{13a}$$

$$\text{where } A_n = 6K_n^G + K_n^c + K_o^c \tag{14}$$

This equation, together with Eqs. (9), (10) and (11) may be used for the determination of the bending moments in symmetrical two-column frames under lateral load and may also be used in the analysis of Vierendeel Girders having parallel and equal flanges. Equation (13) or (13a) is written for each level or story of the frame, and the set of simultaneous algebraic equations thus obtained are solved, by an iterative procedure or by formal algebraic methods, for the joint rotations. The values of the joint rotations then are substituted into Eqs. (9), (10) and (11) to obtain the bending moments in all members.

The transverse displacements, if desired, are obtained in the following manner. Substitution of the numerical values of the joint rotations into Eq. (8) gives the value of R for each story in the same anomalous units as the joint rotations. Since it is probable that the applied moments are not in standard units, it will be necessary to multiply the R-values by a proper factor. For example, if the moments are used in kip-ft units, it will be convenient to multiply the R-values by 12,000 in anticipation of a further adjustment of these quantities to radian measure. Normally relative stiffnesses (K) are used instead of the true stiffnesses (2EI/L) in computing the moments and the R's, and it is also necessary to include a correction for this circumstance before interpreting the R's in radian measure. With these corrections, the incremental transverse displacement of the n-th story becomes

$$\delta_n = kh_n R_n^* \tag{15}$$

in which

$$k = \frac{K}{2EI/L}$$

for any and all members of the frame, and R_n^* is the numerical value of R_n after adjustment for the proper units of moment. The total transverse displacement, Δ, at any level obviously is obtained by summing the δ's from the base of the structure up to and including that level. That is, the total transverse displacements are given by

$$\Delta_n = \sum_{j=1}^{n} \delta_j \tag{16}$$

or by the recursive formula

$$\Delta_n = \Delta_m + \delta_n \tag{17}$$

If an iterative procedure is to be used in solving the set of Eqs. (13), it is convenient to have available simple formulas for good first approximations. These may be obtained directly from Eq. (13) for all stories above the first by setting $\theta_m \approx \theta_n \approx \theta_o$ in that equation. This leads to

$$\theta_n' = \frac{M_m + M_n}{12K_n^G} \tag{18}$$

Somewhat better approximations can be obtained at the first story if the restraint conditions at the bases of the columns are considered. Thus, if the columns are assumed to be built-in at the bottom of the bent, then $\theta_0 = 0$, and if it is also assumed that $\theta_1 \approx \theta_2$, Eq. (13) then yields

$$\theta_1' = \frac{M_1 + M_2}{12K_1^G + 2K_1^c} \qquad \text{(built-in base)} \tag{19}$$

If the columns are assumed to be hinged at the base of the bent, the base may be looked upon as equivalent to a point of contraflexure at precisely the midpoint of a column exactly twice as long as the actual column. Since the applied shear is unchanged, the effective story moment must be doubled. Eq. (18) then yields

$$\theta'_1 = \frac{2M_1 + M_2}{12K_1^G} \qquad \text{(hinged base)} \qquad (20)$$

Illustrative Example

For purposes of illustrating the formulation of the necessary equations and of illustrating the several methods of solution, the bent shown in Fig. 1 is considered. The bent is symmetrical and the columns are assumed to be built-in at the base of the structure.

The data and preliminary calculations are presented in Table I.

The equations are displayed in Table II. Part (A) shows the equation in the form of Eq. (13). Part (B) shows the equations in the alternate, and generally more convenient form of Eq. (13a).

Iterative Method of Solution

The iterative solution is begun with the calculation of approximate values of the joint rotations at alternate stories, using Eq. (18) and, if necessary, Eq. (19) or (20). In the present example, approximate values were obtained for the odd-numbered stories, and are shown in the first line of Table III. At the first story, using Eq. (19)

$$\theta'_1 = \frac{1008 + 924}{12(1) + 2(8)} = 69$$

At the third story, by Eq. (18),

$$\theta'_3 = \frac{660 + 540}{12(1)} = 100$$

The values of θ'_5 and θ'_7 are obtained in a similar manner. It should be noted that it is necessary to apply Eq. (18) only to the alternate stories.

The approximate values thus obtained are substituted into the equations of Table II, either (A) or (B), for the even-numbered rotations, thus yielding the values shown on the second line of Table III. These values are now used to obtain new values of the rotations at the odd-numbered stories, and this process is repeated until the desired degree of convergence is obtained. Four iterations were made in the present problem. It is apparent, however, that four cycles gives considerably greater accuracy than is required by usual engineering standards.

Ordinarily, the slide rule gives sufficient accuracy for engineering purposes. If extreme accuracy is required, a desk calculator may be used; the form of Eq. (13a) being, in fact, especially adapted to this machine.

The initial values and the succeeding approximations are given in Table III. It is to be emphasized that the best rate of convergence is obtained when, for

example, all of the even-numbered rotations are computed before proceeding to the odd-numbered rotations, and vice-versa.

Five cycles of iteration are made in Table III to demonstrate the rapidity of convergence. It is obvious, however, that the values obtained with only two cycles are sufficiently accurate for engineering purposes.

The final column moments are calculated by means of Eqs. (9) and (10) on the basis of the results of the second cycle. The girder moments are calculated by means of Eq. (11). These calculations are made in Table IV. It is also convenient to make such calculations, if desired, directly upon a line drawing of the bent. The final moment-diagram is given in Fig. 2.

It may be noted that the solution is essentially self-checking; it being necessary only to observe that the sum of the moments at each joint is zero.

The lateral deflections at the tops of the various stories are calculated in Table V. Ordinarily this information is not a required part of the stress analysis of a bent, but may be of interest in connection with a vibration analysis[7] or an investigation of dynamic effects.

The columns of the first story have an actual moment of inertia of 15,000 in^4 and a modulus of elasticity of 3×10^6 psi. Hence

$$\frac{2EI}{L} = \frac{2(3 \times 10^6)(15{,}000)}{14 \times 12} = 535.7 \times 10^6 \text{ lb-in.}$$

$$k = \frac{K}{2EI/L} = \frac{8.0}{535.7 \times 10^6} = 14.93 \times 10^{-9} \text{ lb}^{-1} \text{ in}^{-1}$$

The value of k is, of course, constant for all members of the bent.

In connection with the calculation of the joint rotations, the moments were used in units of kip-ft. Accordingly

$$12{,}000\ k = .0001792 \text{ kips}^{-1} \text{ ft}^{-1}$$

Algebraic Solution

The form of Eqs. (13) is such that an exceptionally simple and rapid algebraic solution is also possible, even in the case of many stories. The method consists of eliminating the alternate unknowns, for example, the odd-numbered rotations. The resulting equations will have the same form as the original equations. From this reduced set, the alternate unknowns may be eliminated. This process is continued until a single equation is obtained and the associated rotation evaluated. By reversing the process, the other rotations may be evaluated. The procedure is demonstrated below.

The complete set of equations is

$$\theta_8 = 5.0000 + .2500\theta_7$$

$$\theta_7 = 12.6316 + .1579\theta_8 + .2105\theta_6$$

$$\theta_6 = 21.8182 + .1818\theta_7 + .2727\theta_5$$

$$\theta_5 = 27.6923 + .2308\theta_6 + .3077\theta_4$$

$$\theta_4 = 32.0000 + .2667\theta_5 + .3333\theta_3$$

$$\theta_3 = 35.2941 + .2941\theta_4 + .3529\theta_2$$

$$\theta_2 = 41.6842 + .3158\theta_3 + .3684\theta_1$$

$$\theta_1 = 46.0000 + .3333\theta_2$$

Elimination of the odd-numbered rotations yields

$$\theta_8 = 8.4934 + .0548\theta_6$$

$$\theta_6 = 35.2318 + .0319\theta_8 + .0933\theta_4$$

$$\theta_4 = 62.3846 + .0751\theta_6 + .1434\theta_2$$

$$\theta_2 = 91.1158 + .1213\theta_4$$

Again eliminating alternate joints, the following equations are obtained

$$\theta_8 = 10.4418 + .0051\theta_4$$

$$\theta_4 = 80.0497 + .0025\theta_8$$

Eliminating θ_8 leads to the result

$$\theta_4 = 80.0758$$

Substituting this result into the expression for θ_8 in terms of θ_4 yields

$$\theta_8 = 10.8502.$$

The remaining even-numbered rotations are next computed,

$$\theta_6 = 43.0490$$

$$\theta_2 = 100.829$$

Using the values of the even-numbered rotations in the original set of expressions for the odd-numbered rotations yields

$$\theta_7 = 23.4067$$

$$\theta_5 = 62.2673$$

$$\theta_3 = 94.4269$$

$$\theta_1 = 79.6063$$

Iterative-Algebraic Solution

In certain cases it may be advantageous to use a combination of algebraic and iterative steps. For instance, if the columns are extremely stiff in comparison with the girders, the method outlined and demonstrated below converges, as will be seen, much more rapidly than a completely iterative solution. The method may be preferred, also, when the bent contains many stories.

If the alternate unknowns are eliminated from the original set of equations, a set is obtained in which the coupling between the remaining adjacent unknowns is greatly reduced as compared with the coupling between the adjacent unknowns. Therefore this set of equations will converge much faster than the original set. In addition, only half as many unknowns are involved in the iterative process. If the coupling between the unknowns is still too great, a second cycle of eliminations may be made, and the iterative method applied to the then resulting equations.

In the illustrative example of the previous section, eliminating the odd-numbered joints yields the following set of equations,

$$\theta_8 = 8.4934 + .0548\theta_6$$

$$\theta_6 = 35.2318 + .0319\theta_8 + .0933\theta_4$$

$$\theta_4 = 62.3846 + .0751\theta_6 + .1434\theta_2$$

$$\theta_2 = 91.1158 + .1213\theta_4$$

The coefficients on the right hand sides of these equations are seen to be considerably smaller than those in the original set of eight equations.

An iterative solution may be made of these equations. Assuming, for the purpose of obtaining a first approximation, that $\theta_8 = \theta_6 = \theta_4$ in the second equation leads to

$$\theta_6 = 35.2318 + .1252\theta_6$$

which yields the first approximation

$$\theta_6 = 40.3$$

Similarly, setting $\theta_4 = \theta_2$ in the last equation gives the first approximation

$$\theta_2 = 102.7$$

Substituting these values into the expressions for θ_8 and θ_4 yields

$$\theta_8 = 10.70$$

$$\theta_4 = 80.13$$

A second cycle gives

$$\theta_6 = 43.05$$

$$\theta_2 = 100.84$$

and, using these values yields

$$\theta_8 = 10.85$$

$$\theta_4 = 80.06$$

The odd-numbered rotations may now be calculated by means of the original equations since they are expressed entirely in terms of the even-numbered rotations.

Obviously even faster convergence may be had from the equations which are obtained by two cycles of algebraic elimination.

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TABLE I

CONSTANTS

STORY	1	2	3	4	5	6	7	8
h (ft.)	14	14	12	12	12	12	12	12
Conc. Load (kips)	6	11	10	10	10	10	10	5
S = Shear (kips)	72	66	55	45	35	25	15	5
$M_n = S \times h$ (Kip-ft.)	1008	924	660	540	420	300	180	60
$\frac{M_n + M_o}{2}$	966	792	600	480	360	240	120	30
K_n^G	1	1	1	1	1	1	1	0.75
K_n^c	8	7	6	5	4	3	2	1.5
$A_n = 6K_n^G + K_n^c + K_o^c$	21	19	17	15	13	11	9.5	6

TABLE II

EQUATIONS

	(A)	(B)
6	$\theta_8 = 30 + 1.5\theta_7$	$\theta_8 = 5.0000 + .2500\theta_7$
9.5	$\theta_7 = 120 + 1.5\theta_8 + 2\ \theta_6$	$\theta_7 = 12.6316 + .1579\theta_8 + .2105\theta_6$
11	$\theta_6 = 240 + 2\ \theta_7 + 3\ \theta_5$	$\theta_6 = 21.8182 + .1818\theta_7 + .2727\theta_5$
13	$\theta_5 = 360 + 3\ \theta_6 + 4\ \theta_4$	$\theta_5 = 27.6923 + .2308\theta_6 + .3077\theta_4$
15	$\theta_4 = 480 + 4\ \theta_5 + 5\ \theta_3$	$\theta_4 + 32.0000 + .2667\theta_5 + .3333\theta_3$
17	$\theta_3 = 600 + 5\ \theta_4 + 6\ \theta_2$	$\theta_3 = 35.2941 + .2941\theta_4 + .3529\theta_2$
19	$\theta_2 = 792 + 6\ \theta_3 + 7\ \theta_1$	$\theta_2 = 41.6842 + .3158\theta_3 + .3684\theta_1$
21	$\theta_1 = 966 + 7\ \theta_2$	$\theta_1 = 46.0000 + .3333\theta_2$

TABLE III

CALCULATION OF JOINT ROTATIONS BY ITERATION

STEP	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8
θ'	69		100		60		20	
First Approximation	78.9	98.7	94.0	81.3	62.4	41.8	23.0	10.0
Second Approximation	79.48	100.44	94.26	79.97	62.23	43.02	23.39	10.75
Third Approximation	79.58	100.73	94.38	80.01	62.24	43.04	23.41	10.85
Fourth Approximation	79.60	100.81	94.42	80.06	62.26	43.05	23.41	10.85
Fifth Approximation	79.61	100.83	94.43	80.08	62.27	43.05	23.41	10.85

TABLE IV

COMPUTATION OF GIRDER AND COLUMN MOMENTS, TWO CYCLES OF ITERATION

Story, n	1	2	3	4	5	6	7	8
K_n^G	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.75
K_n^c	8.0	7.0	6.0	5.0	4.0	3.0	2.0	1.5
$\frac{M_n}{4}$	252.0	231.0	165.0	135.0	105.0	75.0	45.0	15.0
θ_n	79.5	100.4	94.3	80.0	62.2	43.0	23.4	10.8
$\theta_n - \theta_m$	79.5	20.9	- 6.1	- 14.3	- 17.8	- 19.2	- 19.6	- 12.6
$\frac{K_n^c}{2}(\theta_n - \theta_m)$	318.0	73.2	- 18.3	- 35.8	- 35.6	- 28.8	- 19.6	- 9.5
$M_{nm}^c = \frac{M_n}{4} - \frac{K_n^c}{2}(\theta_n - \theta_m)$	- 66.0	157.8	183.3	170.8	140.6	103.8	64.6	24.5
$M_{nm}^c = \frac{M_n}{4} + \frac{K_n^c}{2}(\theta_n - \theta_m)$	570.0	304.2	146.7	99.2	69.4	46.2	25.4	5.5
$M_n^G = -3K_n^G\theta_n$	-238.5	-301.2	-282.9	-240.0	-186.6	-129.0	- 70.2	- 24.3
CHECK: $M_n^G + M_{nm}^c + M_{no}^c$	- 0.3	3.3	- 0.4	0.2	0.2	0.2	- 0.1	0.2

TABLE V

LATERAL DEFLECTIONS AT TOP OF EACH STORY

STORY	1	2	3	4	5	6	7	8
h_n (in.)	168	168	144	144	144	144	144	144
K_n^c	8.0	7.0	6.0	5.0	4.0	3.0	2.0	1.5
M_n (kip-ft.)	1008	924	660	540	420	300	180	60
θ_n	79.5	100.4	94.3	80.0	62.2	43.0	23.4	10.8
$\frac{M_n}{12K_n^c}$	10.5	11.0	9.2	9.0	8.8	8.3	7.5	2.2
$\frac{1}{2}(\theta_n + \theta_m)$	39.7	90.0	97.4	87.1	71.1	52.6	33.2	17.1
$R_n = \frac{M_n}{12K_n^c} + \frac{1}{2}(\theta_n + \theta_m)$	50.2	101.0	106.6	96.1	79.9	60.9	40.7	19.3
$\delta_n = 0.0001792 R_n h_n$ (in.)	1.51	3.04	2.75	2.48	2.06	1.57	1.05	0.50
$\Delta_n = \sum_1^n \delta_i$ (in.)	1.51	4.55	7.30	9.78	11.84	13.41	14.46	14.96

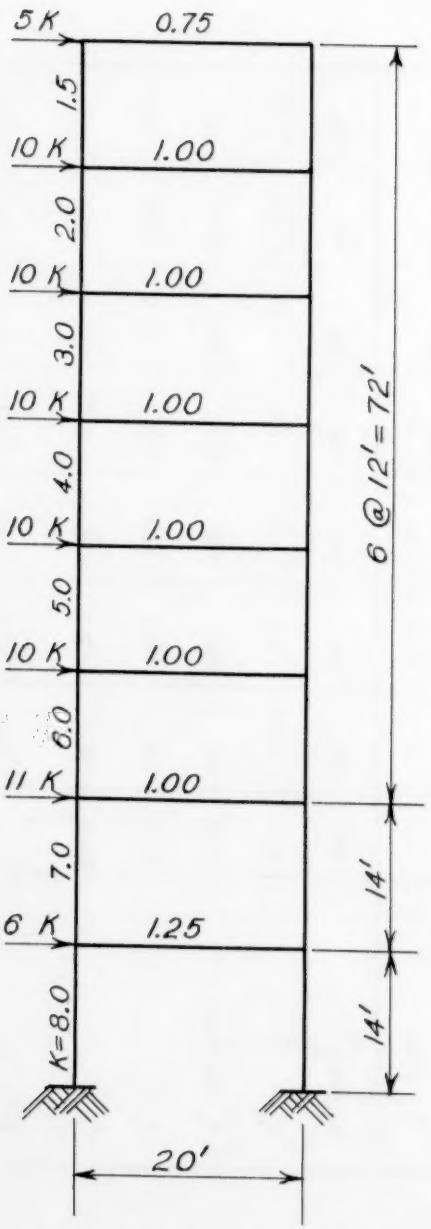
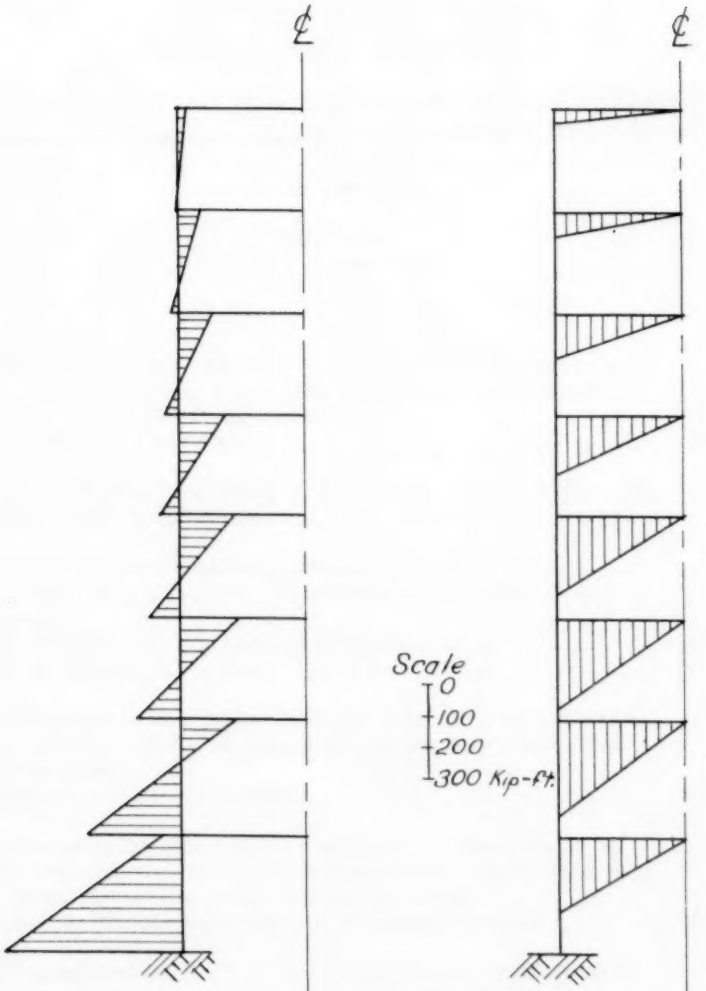


Fig. 1. Bent of Illustrative Problem



Moments are plotted on tension side of member.

Fig. 2. Column and Girder Moments, Two Cycles

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SHELLS OF DOUBLE CURVATURE[a]

Closure by A. L. Parme

A. L. PARME,[1] M. ASCE.—The pertinent and informative comments made by the various contributors on a subject that promises to be of great interest in the immediate future are deeply appreciated. In his discussion T. Au develops formulas facilitating to some extent the determination of the effect of lateral loads by the use of an integrating factor. This simplification does not, however, ease the problem of satisfying the boundary conditions. For example, the shear created in a hyperbolic paraboloid shell due to a uniform lateral load (w_x) acting in the x direction is by equation (7b) expanded to include the effect of a lateral force

$$2 S_p \frac{h}{ab} - \left(\frac{hy}{ab}\right) w_x = 0$$

or

$$S_p = \frac{w_x y}{2}$$

Substituting this in eq. 5 modified to include lateral surface load

$$\frac{\partial T_{xp}}{\partial x} + \frac{3w_x}{2} = 0$$

which on integrating yields

$$T_{xp} = \frac{-3w_x X}{2} + C$$

With only one constant of integration available to satisfy edge conditions for lateral forces at two boundaries it is apparent that the membrane analysis which implies only direct axial and tangential shearing in a shell is applicable or valid only if suitable reactions occur along one of the boundaries. If this latter condition is not satisfied as in the case of free standing umbrella type hyperbolic paraboloids it follows that resistance to uniform lateral load must be supplied by the presence of radial shears and bending moments in the

a. Proc. Paper 1057, September, 1956, by A. L. Parme.

1. Manager, Structural and Railways Bureau, Portland Cement Assoc., Chicago, Ill.

shell. It can be shown that for a few loading cases, membrane analysis is sufficient. But in most practical cases such as that reviewed recourse must be made to an analysis which includes bending moments. So far there is no simple way of treating the problem of bending in hyperbolic paraboloid shells.

It is true that generalized differential equations expressing the equilibrium of internal forces including bending moments in a shell have been derived. But direct rigorous solution of these differential equations has not progressed to a point where it can be applied readily. Quite frequently approximate solutions based on limiting modes of behavior are employed. For example, in the umbrella type of shell the thickness of the shell in the valley near the column is of such magnitude that the entire bending moment can be resisted in this local region without any difficulty. The dissipation of the moment into the shell is then approximated on the basis of several limiting conditions of radial shear distribution.

Mr. Au points out rightly so to the misleading statement following eq. 10. This has been corrected.

Mr. Pagon points out the need for more analytical information on the subject, especially with regard to the effect of lateral loads. As previously mentioned, an investigation of the stresses created by lateral loads requires bending analysis of the hyperbolic paraboloid shell. The magnitude of such a study is beyond the scope of the present paper.

The question raised by Mr. Pagon on the degree of flatness which can be used and still have the membrane analysis valid for the hyperbolic paraboloid depends to a large extent on the magnitude of the secondary bending moments caused by axial strains. The analysis presented is based on a satisfaction solely of the equilibrium of forces, and no attention is given to the compatibility between strains and stresses. For the usual rise, say h/a or h/b = 5 the effect of axial strains is unimportant and can be safely ignored. However, when the ratio h/a decreases the effect of axial strains commences to exert a dominant influence on the behavior of the shell. The departure in behavior from that indicated by the simple membrane analysis in a flat shell is analogous to that occurring in a two hinged parabolic arch subject to uniform load as the ratio of rise to span decreases. For very flat parabolic arches, it can be shown that if rib shortening effect ie: that is axial deformations, is included in the analysis, the horizontal component of the reaction for a given span decreases as the ratio of rise to span decreases. With no rise, the horizontal component decreases to zero, and thus the secondary bending due to axial strains approaches as a limiting value the simple beam bending moment.

Since the structural action of a hyperbolic paraboloid shell is due to the fact that its curved surface resists the load by two sets of parabolic arches perpendicular to each other as shown in Fig. 3 some insight into the effect of curvature can be obtained by examining a strip parallel to the arches as a free body. If the shearing and normal forces on the two opposite faces are ignored, and it is assumed that the end of the arches are not free to move, then the secondary bending moments due to lack of curvature can be determined as for an arch. The result of such a study is presented in Fig. 12 for various ratios of ht/ab.

The secondary bending moment at various distances from the corner designated by the dimensionless quantity x/t is expressed in terms of the simple beam bending moments occurring in a strip of length L. Fig. 12 indicates that at the corner because the ratio of rise to span approaches 0, the load is

carried entirely by beam action contrary to what is expected by membrane theory. For strips farther away from the corner the secondary moment decreases. The rate of this decrease is a function of ht/ab. The larger the ratio of ht/ab, the more rapid is the decrease in the magnitude of the secondary moments. The usual value of ht/ab for umbrella type is about .010. Assuming that the thickness is about 3 inches, the secondary moment becomes unimportant about 5 ft. from the corner.

This chart brings out another important characteristic which has been observed on some of the shells built. It is seen that at the corner the load is carried mainly by ordinary beam action. Hence the load is transmitted to the edge beams principally by radial shears. The edge beams in this vicinity are thus loaded vertically and act as cantilevers for a small portion of their length. The edge beams in this vicinity should consequently not only be designed for the tension computed by membrane theory but should also be deepened to prevent excessive deflection, and reinforced for negative moment. This is especially desirable when the edge beam is upturned.

Because the value of L increases linearly in proportion to the distant from the corner, it is more expedient to show the effect of axial strains in terms of the secondary flexural stresses created. Such values are plotted in Fig. 13. This chart brings into sharper relief the importance of curvature on the magnitude of the secondary stresses. For an umbrella type of hyperbolic paraboloid subject to 72 psf load with ht/ab = .01, the secondary stresses occur at x/t = 13 and are

$$f_c = \frac{110x72}{144x2} = 27 \text{ psi}$$

For the analysis of various types of shells with different types of boundary requested by Mr. Pagon, no generalized solution can be given. It is for this reason that equations based on finite differences were presented.

Mr. Bannerjee's preference to deal directly with the stresses instead of the stress function F is in accord with the author's experience and suggested alternate finite difference eq. 43. His suggested modification of the formulas to cover the case of oblique shape is a welcomed addition.

Dr. Salvadori's application of the well known solution for cylindrical surfaces to the determination of the bending stresses occurring near the arches in the elliptical paraboloid represents the only practical approach towards estimating the bending in shells of double curvature. But in employing this approximation, the difference between the idealized state which form the basis of the formulas presented by Dr. Salvadori and the actual conditions must be borne in mind. The magnitude of the bending moments presented in Salvadori's equation (2) and (5) are those occurring in an infinitely long cylinder subject to a radial line load applied uniformly about the perimeter. The formulas are also valid for cylinders subject to a radial line load varying sinusoidally as sine($n\pi$) about the perimeter as long as $n\sqrt{t/r}$ is less than 0.50. In both cases, it is implied that the resistance to the line load or imposed deformation is developed by compression in the adjacent elements acting as fully continuous rings. When the rings are not continuous, their stiffness is greatly reduced and hence the applicability of the formulas is very questionable. Thus while Salvadori's equations (2) and (5) may predict with a fair degree of accuracy the structural behavior near the crown, because the horizontal strips are continuous they may exaggerate the moment near the corner.

It should also be pointed out that the equations assume that edges of the shell are fully restrained against angular rotation. This is not usually the case since the arches have little torsional stiffness. With no restraint, the maximum moment occurs some distance from the edge and its value is approximately one third of that indicated by Salvadori's eq. 2.

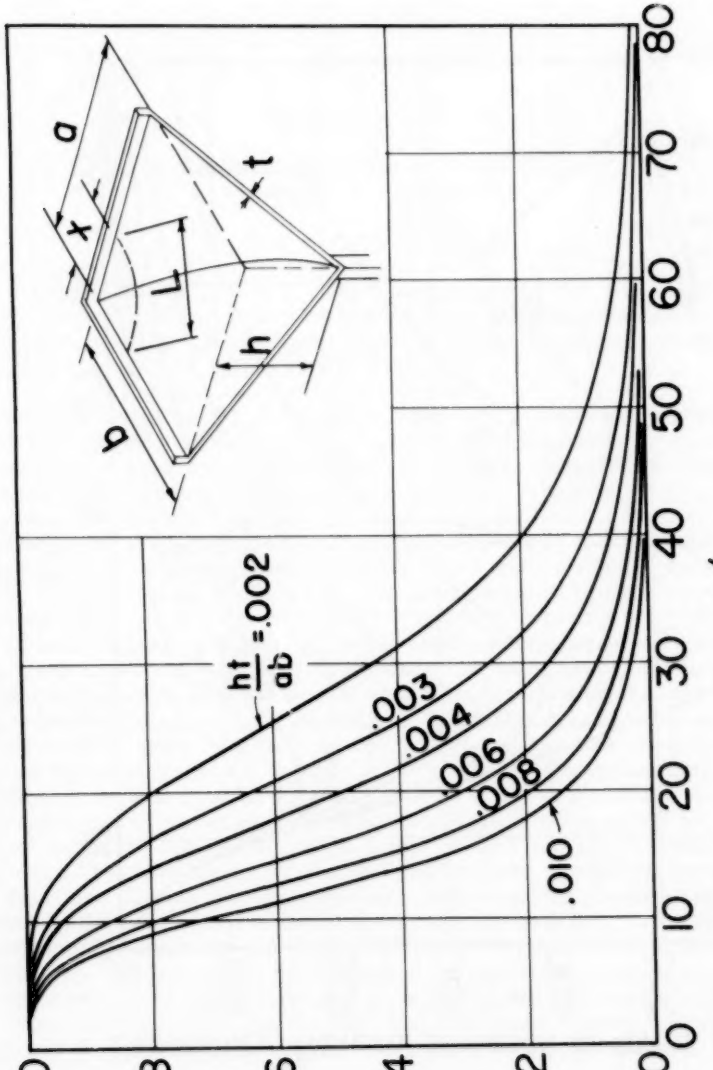
Secondary moment in terms of $\frac{WL^2}{8}$
1.0
.8
.6
.4
.2
0
0
10
20
30
40
50
60
70
80
x/t
$\frac{ht}{ab}$ =.002
.003
.004
.006
.008
.010
a
b
x
L
h
t

Fig. 12

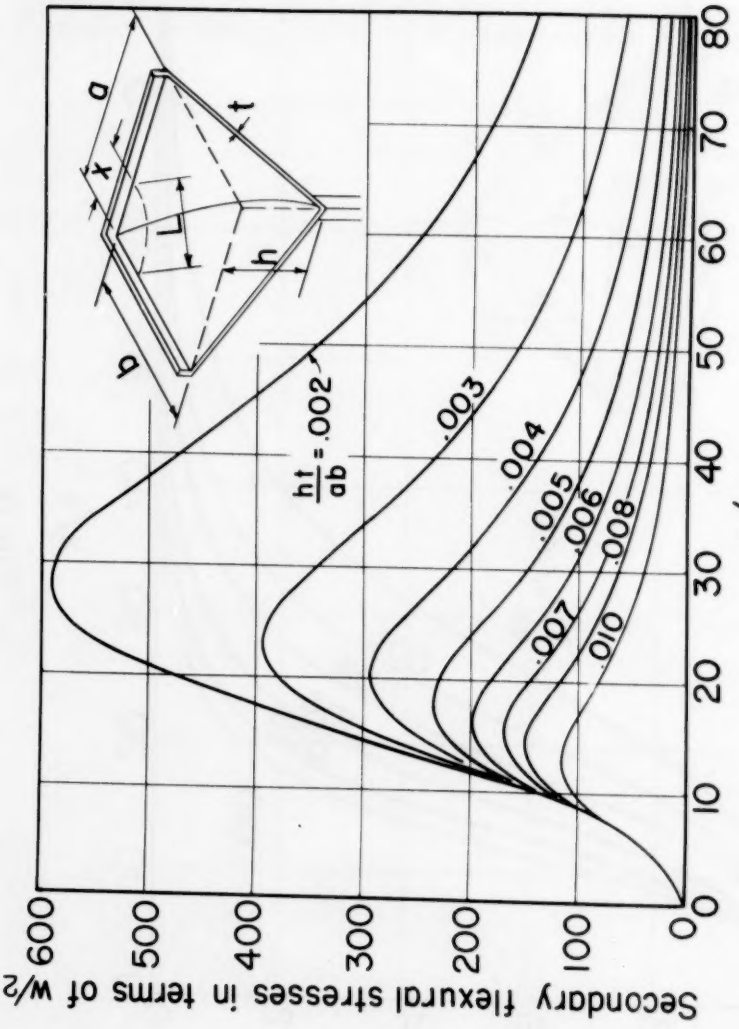
Secondary flexural stresses in terms of w/2
600
500
400
300
200
100
0
0
10
20
30
40
50
60
70
80
x/t
$\frac{ht}{ab}$ = .002
.003
.004
.005
.006
.008
.007
.010
a
x
L
h
t
b

Fig. 13

ANALYSIS OF MULTI-STORY BUILDING FRAMES[a]

Corrections to discussion by E. Neil W. Lane
Closure by T. F. Hickerson

CORRECTION—In E. Neil W. Lane's discussion of Proc. Paper 1233—which appeared in Proc. Paper 1442 (November, 1957)—the mention on page 16, line 5 of Mitchell's elastic traverse should be changed to Stewart's (Ralph W. Steward, M. ASCE) elastic traverse.

T. F. HICKERSON,[1] M. ASCE.—The conventional methods of dealing with joint and story equilibrium in a building frame—and the equations resulting therefrom have long been established and generally recognized as common knowledge. In 1929 the writer used these equations repeatedly while teaching a graduate course in structures. The reference text(1)—still a classic—introduces the slope-deflection theorem and applies it to the complete analysis of a 5-story building frame. The values of θ and x leading to the calculation of the column and girder moments are determined by the solution of 5 equations.

To find independent values of θ and x without solving equations simultaneously was a problem that intrigued the writer for many years. Various approximate formulas were derived by simple algebraic reductions, but the assumption "that all joints in the nth floor and in the adjacent upper and lower floors have the same rotation" provided the preliminary step needed to round out the solution. This hint, obtained in 1956 from the Maney-Goldberg Method,(2) leads to Equations (18) and (19), for which sole credit was given to Mr. Goldberg. However, upon investigation, it appears that credit for this "idea" is due the late Mr. George A. Maney(3)—in a sort of professor-student relationship—where Maney had the idea and Goldberg carried out the scheme.

Equations (18) and (19)—in combination with equations developed independly by the writer-lead to the key Equations (24) and (27). The other equations including Equations (1), (2), (9), (10), (15), (16) and (17) were derived independly. Aside from the substitution in the preliminary Equations (24) and (27), the procedure differs from that presented by Mr. Goldberg, as indicated by Equations (4), (5), (6), (7), (9), etc. This difference is more fundamental than that of convergence technique, as suggested by Mr. Williamson.

Introduction of the factors $\frac{K}{\Sigma K}$ and $\frac{K}{2\Sigma K}$, the former giving coefficients of x in the first convergence, the latter giving carry-over rotations, are features of simplicity and directness not possessed by the Goldberg Method. The factors $\frac{K}{2\Sigma K}$ also enter into the second convergence.

a. Proc. Paper 1233, May, 1957, by T. F. Hickerson.

1. Formerly Prof. of Civ. Eng., Univ. of North Carolina, Chapel Hill, N. C., (Ret).

A derivation omitted for the sake of brevity will now be presented. Let $E\theta_N$ (see Fig. 4) apply to the rotation at Joint N while r_o, r_1, r_2, r_3 equal the ratios of the rotations at the far ends 0, 1, 2 and 3 relative to that at the near end (N).

For equilibrium at Joint N, $\Sigma M = 0$, then by the slope-deflection theorem, we have

$$2EK_1[2\theta N + r_o\theta_N - 3x_1] + 2EK'[2\theta_N + r_1\theta_N] + 2EK_2[2\theta_N + r_2\theta_N = 3x_2] + 2EK''[2\theta_N + r_3\theta_N] = 0 \quad (a)$$

Dividing by 2E, and reducing, we get

$$\theta_N[(2 + r_o)K_1 + (2 + r_1)K' + (2 + r_2)K_2 + (2 + r_3)K''] = 3K_1x_1 + 3K_2x_2 \quad (b)$$

If the rotations at the far end are the same as that at the near end (N), then $r_o = r_1 = r_2 = r_3 = 1$, hence Equation (b) reduces to

$$\theta_N = \frac{K_1x_1 + K_2x_2}{\Sigma K_J} = (\frac{K_1}{\Sigma K_J})x_1 + (\frac{K_2}{\Sigma K_J})x_2, \quad (c)$$

which is the same as Equation (5).

If the column is fixed at the base [that is, $r_o = 0$], while $r_1 = r_2 = r_3 = 1$, Equation (b) becomes

$$\theta_N = \frac{3K_1x_1 + 3K_2x_2}{2K_1 + 3K' + 3K_2 + 3K''} = \frac{K_1x_1 + K_2x_2}{\Sigma K_J - \frac{1}{3}K_1} \quad (d)$$

Equation (d) establishes Equation (7) which plays a role in the development of Equation (27).

Mr. Sobotka's discussion is constructive and a valuable addition to the paper for which he is due an expression of thanks.

It is gratifying that Mr. Lane sees the advantage of the $\frac{K}{2\Sigma K}$ factors and the practical parallel to the Cross procedure. He is to be thanked for the painstaking solution of Example 2 by the Cross Method, which, however appears formidable when applied to a tall building frame, whereas the $\frac{K}{\Sigma K}$ and $\frac{K}{2\Sigma K}$ factors would tend to become constant while proceeding upward from the bottom.

REFERENCES

1. Part II, "Statically Indeterminate Structures" by Johnson, Bryan and Turneaure, John Wiley and Sons, Inc.

2. "Analysis of Statically Indeterminate Structures" by Parcel and Moorman, John Wiley and Sons, Inc.
3. "Simplified Methods for the Analysis of Multiple Joint Rigid Frames" by George A. Maney and John E. Goldberg, Bulletin, Northwestern Univ., 1932.

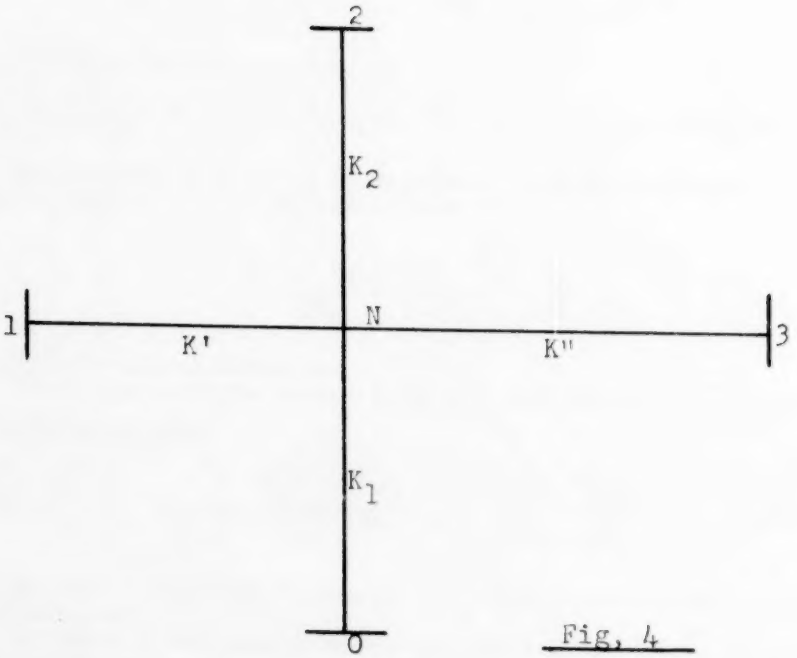

Fig. 4

BEAM DEFLECTION IN BRIDGES DESIGNED FOR CONTINUITY[a]

Closure by Guillermo Villena

GUILLERMO VINNENA,[1] A.M. ASCE.—The writer expressed sincere thanks to Mr. Sobotka for his fine contribution to this paper. In checking his formulas for the first part of his discussion (constant moment of inertia) the writer emphasizes that if only one point is desired, the maximum deflection at the end span, then Table I on page 1442-21 should be more realistic as far as establishing the tabular solution to his equations 6, 7 and 8.

$$C_o^3 - (3C_o^2 - 1)K - 1/4 = 0 \qquad (6)$$

Where:

$$K = 1/2\left(1 - \frac{M_R}{4M_S}\right) \qquad (7)$$

Then:

$$K = \frac{C_o^3 - 1/4}{3C_o^2 - 1} = \frac{1/4 - C_o^3}{1 - 3C_o^2} \qquad (8)$$

According to the formulas above, $\frac{M_R}{M_S}$ was made the dependent variable of C_o in order to simplify the solutions of an otherwise cubic equation. It would be better, however, if $\frac{M_R}{M_S}$ is made the independent variable since its value will be precisely known when finding C_o for which Δ_x is maximum.

Naturally if we are to plot the camber diagram, then the maximum deflection will show up. Since most computation will be done with slide rule anyway, the small difference will be too insignificant to be appreciated.

However, it is worthwhile remembering that in bridge design where the first and last support are usually free, it is of upmost importance to keep these reactions acting downwards at all times, including the case where there is no live load.

Generally, for the end span $R_L = \frac{wL_1}{2} = \frac{M_R}{L_1}$

By this equation R_L should always be positive.
For the extreme case when $R_L = 0$

a. Proc. Paper 1234, May, 1957, by Guillermo Villena.
1. Project Engr., Alfred Le-Feber & Associates, Cincinnati, Ohio.

$$\frac{wL_1}{2} - \frac{M_R}{L_1} = 0 \qquad \frac{wL_1}{2} = \frac{M_R}{L_1}$$

or

$$\frac{wL_1^2}{8} = \frac{M_R}{4} \qquad \frac{M_R}{M_S} = 4$$

On the other hand when $\frac{M_R}{M_S} = 0$, then $M_R = 0$, and the structure becomes simply supported, a case for which $C_o = 1/2$.

When $\frac{M_R}{M_S} = 4$, then $K = 0$ (this value is not obtained when $C_o = 1/2$ as shown in Table I) and thus from Mr. Sobotka's equation (8)

$$C_o^3 = 1/4 \qquad C_o = \frac{1}{\sqrt[3]{4}} = 0.63$$

This will determine the location of maximum negative deflection when $R_L = 0$.

THE BEHAVIOR OF ONE-STORY REINFORCED CONCRETE SHEAR WALLS[a]

Closure by Jack R. Benjamin and Harry A. Williams

J. R. BENJAMIN,[1] A.M. ASCE and HARRY A. WILLIAMS,[2] M. ASCE.—The writers wish to thank Mr. Matteson for his review of their paper. His remarks concerning crack formation and propagation in reinforced concrete are appreciated.

It was noted in the original paper that studies based on the lattice analogy method of analysis were made for a number of models. Principal stress patterns were determined and contours plotted for several cases. An example is shown in Fig. 40. (The results of a simplified approach are shown in Fig. 41 for comparison.) In this case, C-1 and C-5 were identical specimens with 3.3% column steel and no panel reinforcing. However, the compressive strength of the concrete of C-1 was 1.57 times that of C-5.

An initial crack occurred near the edge of the most highly stressed region where it was free to propagate rather than in the most highly stressed corner. (Fig. 40). It is probable that the other branches of crack No. 1 shown in Figs. 42 and 8 formed after the initial crack as a result of re-distribution of stress in the panel. General failure occurred along crack No. 2 which was a continuation of small branches of crack No. 1. The main point is that the cracks which started in the corner did not progress because of resistance from the column and foundation. The increased thickness seems to have been sufficient for C-5; the column steel had the necessary retarding effect for C-1.

Study of all of the reinforced concrete walls shows no relationship between panel reinforcing and location of first crack. Local strengthening and weakening as well as many other variations were studied. The reinforcing is only effective after cracking begins, not before. This statement is true with respect to the columns as well as the panel. Changes in the column reinforcing did not change the mode of panel cracking provided tensile column failure could not occur. The tests indicate that the tensile column-foundation junction tends to fail first. If sufficient column steel is provided to carry the tensile load after the tension column concrete cracks, the wall panel shows the first visual cracking. If insufficient tensile column steel is provided to carry the tension in the tension column when the concrete cracks, the first crack observed is at the junction of tension column and foundation. This implies that the tension column cracks first with all specimens but the crack does not open sufficiently to observe it because of the steel across the section.

The bending stiffness of the loading beam and columns has no influence on first cracking in shear walls. The larger their areas, the more nearly a pure

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a. Proc. Paper 1254, May, 1957, by Jack R. Benjamin and Harry A Williams.
1. Assoc. Prof. of Civ. Eng., Stanford Univ., Stanford, Calif.
2. Prof. of Civ. Eng., Stanford Univ., Stanford, Calif.

shear condition exists in the wall panel. This is observed in many of the tests.

From the standpoint of seismic design, the useful range for design is limited to a damage criterion. In general, the walls are so badly cracked at ultimate load that their usefulness is at an end even for blast resistant construction where large cracks are expected. It is questionable if seismic loads should be allowed much beyond the appearance of the first crack. While the structure may stand and actually be perfectly usable, its appearance may be such as to require extensive costly repairs.

The linear deflection range was studied extensively to discover the influence of reinforcing steel on the load-deflection curve. Careful comparison of test results showed no correlation between amount of steel and rigidity until the wall cracked. Thus, the deflection formula is based on a section of pure unreinforced concrete. Any such study must be based on statistical methods. With shear walls, normal variation in behavior makes the separation of small influences most difficult, if not impossible. Thus if a simple formula gives an answer that is as statistically sound as a more complex approach, the authors believe the simple formula is best.

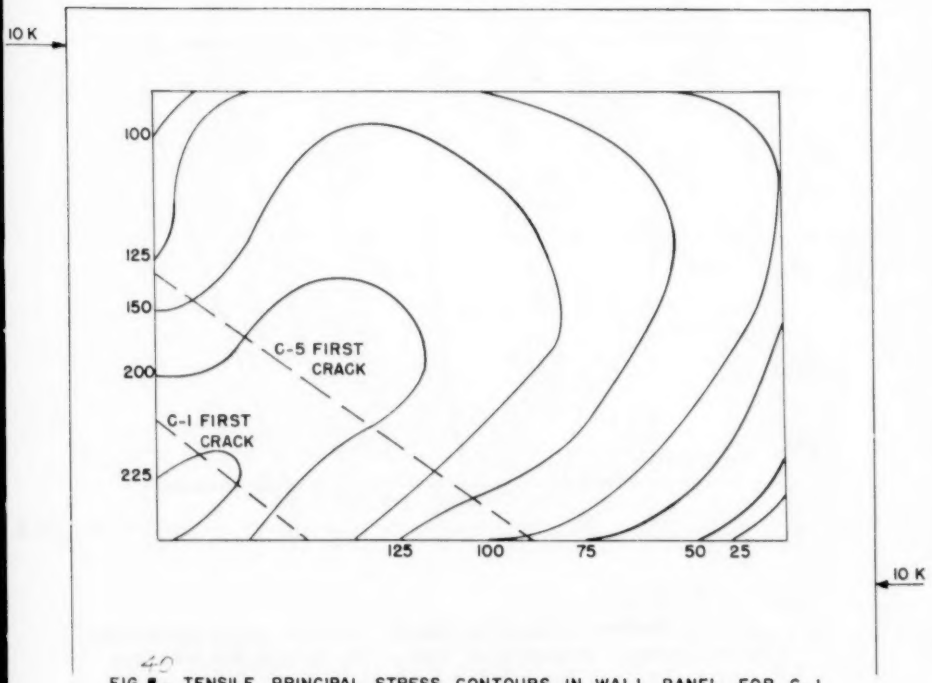

FIG. 8 - TENSILE PRINCIPAL STRESS CONTOURS IN WALL PANEL FOR C-1 AND C-5. SOLUTION BY LATTICE ANALOGY. STRESSES IN PSI. FIRST WALL CRACKS SHOWN.

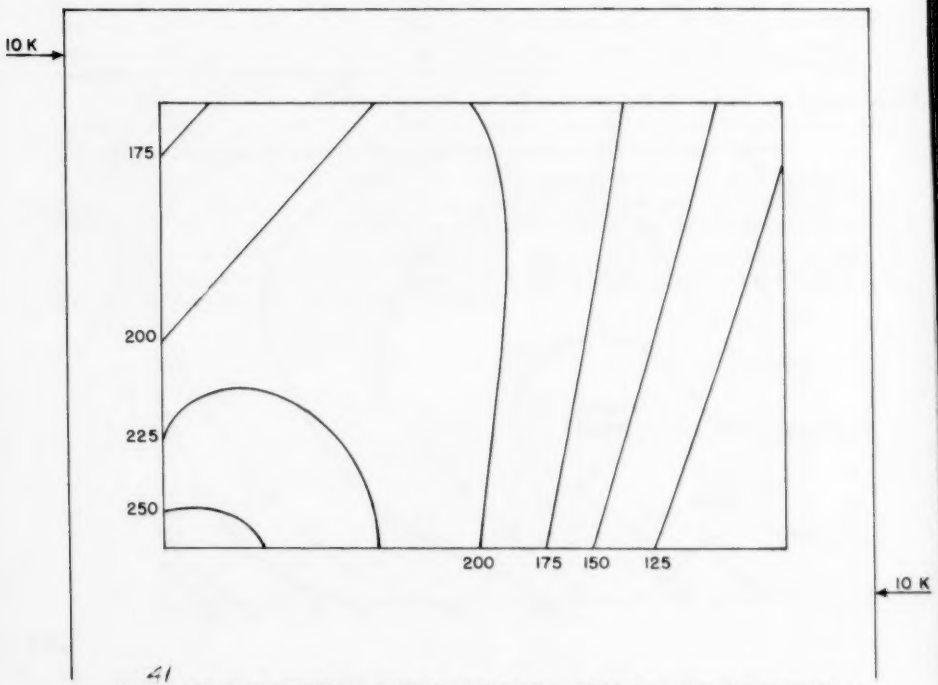

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FIG. 6 - TENSILE PRINCIPAL STRESS CONTOURS, COMPUTED USING ELEMENTARY BEAM THEORY. STRESSES IN PSI., FOR WALLS C-1 AND C-5.

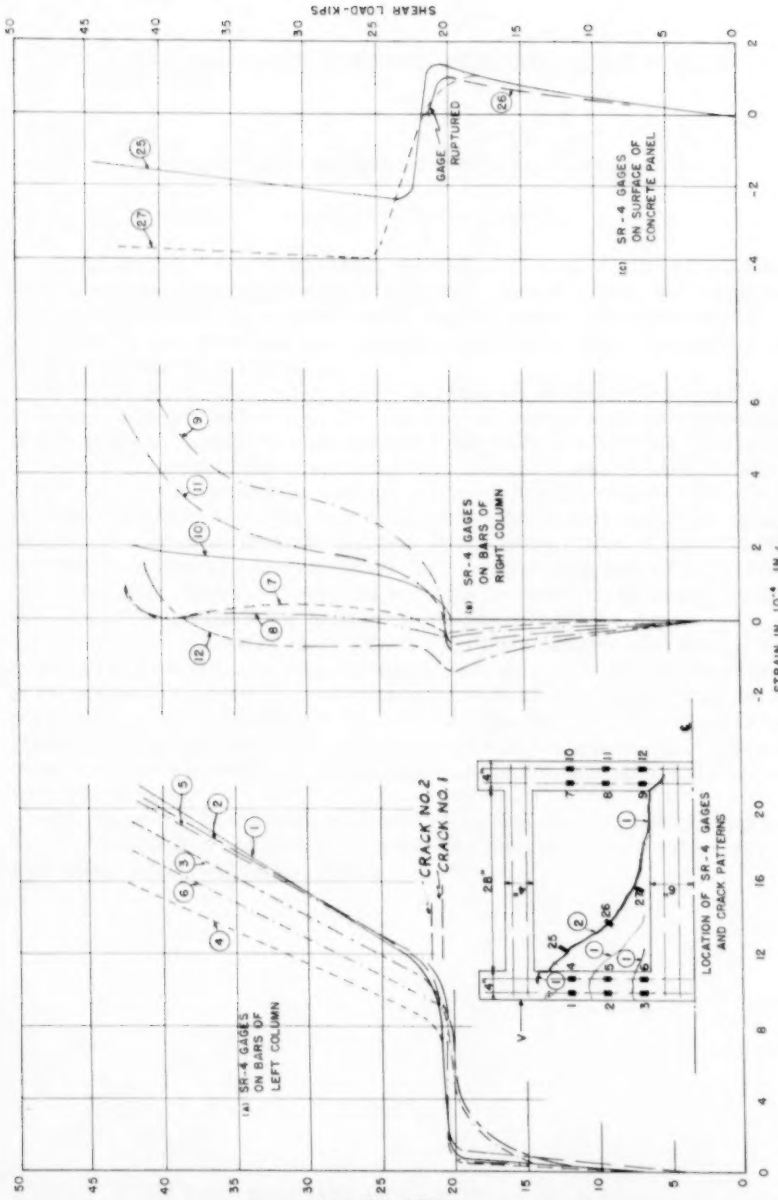

SHEAR LOAD-KIPS
STRAIN IN 10^{-4} IN./IN.
(a) SR-4 GAGES ON BARS OF LEFT COLUMN
(b) SR-4 GAGES ON BARS OF RIGHT COLUMN
(c) SR-4 GAGES ON SURFACE OF CONCRETE PANEL
CRACK NO.2
CRACK NO.1
GAGE RUPTURED
LOCATION OF SR-4 GAGES AND CRACK PATTERNS

FIG. 13.- REINFORCING BARS AND CONCRETE STRAINS FOR SPECIMEN C-1

LATERAL LOAD DISTRIBUTION TEST ON I-BEAM BRIDGE[a]

Closure by Ardis White and William B. Purnell

ARDIS WHITE,[1] and WILLIAM B. PURNELL,[2] A.M. ASCE.—The authors wish to express their appreciation to Messrs. Naruoka, Wei, and Balog for their discussions of the original paper, as well as for the submission of a great deal of analytical and experimental verification of the results and conclusions presented in that paper.

Mr. Naruoka calls attention to the fact that the behavior of a flexible shear connector changes as the ultimate load of a composite beam is approached. In this respect, it may be noted that the loads placed on highway bridges under normal use would never approach the ultimate loads the bridge could support. It seems reasonable to assume that even flexible shear connectors of the type discussed in the paper would always develop full composite action in positive moment regions under normal design loads. (Data obtained using a 100,000 lb. pavement concrete mixer gave results in agreement with those presented in the paper. These data were not included in the paper because the mixer was available for only one load application.)

Mr. Wei has presented much information from his excellent thesis, "Effects of Diaphragms in I-Beam Bridges," and his analytical computations regarding lateral transfer of total moments check almost exactly with the experimental results obtained by the authors. Mr. Wei is correct in calling attention to the fact that the results presented by the authors were obtained from a test on a specific structure, and that the conclusions should be read with this in mind.

Mr. Balog has presented the results of a great deal of fruitful experimental work. Conclusions and results presented by the authors are essentially in agreement with those of Mr. Balog.

a. Proc. Paper 1255, May, 1957, by Ardis White and William B. Purnell.
1. Asst. Prof. of Civ. Eng., Univ. of Houston, Houston, Tex.
2. Assoc. Prof. of Civ. Eng., Univ. of Houston, Houston, Tex.

ALUMINUM APPLICATIONS FOR HIGHWAY BRIDGES[a]

Closure by J. M. Pickett

J. M. PICKETT,[1] J.M. ASCE.—The author wishes to thank Mr. S. K. Ghaswala for the interest shown in the subject of aluminum bridges, and specifically the interesting points which he contributes in his discussion.

In December, 1957, the Iowa State Highway Department awarded a contract to Jensen Construction Company, United Contractors, of Des Moines, Iowa, for a 220 ft. long, four-span continuous aluminum highway bridge to be erected near Des Moines. This bridge was designed by Mr. Ned L. Ashton of Iowa City, Iowa. The aluminum components will be fabricated by the Pullman-Standard Company of Chicago. It is expected that the structure will be completed by the end of this summer.

Other aluminum bridge projects are being designed and considered at this time. It may be assumed that there will be considerably more activity in this field by the end of the year.

a. Proc. Paper 1312, July, 1957, by J. M. Pickett.

1. Development Engr., Sales Development Div., Aluminum Co. of America, New Kensington, Pa.

LOAD FACTORS FOR PRESTRESSED CONCRETE BRIDGES[a]

Closure by T. Y. Lin

T. Y. LIN,[1] M. ASCE.—The writer studied Mr. Lane's discussion with great care, but was still unable to grasp the meaning of all the points brought up in the discussion. Apparently, the writer had no disagreement with the ideas and facts expressed in the paper, but simply tried to emphasize certain practical considerations which could affect the safety of a structure, besides the load factor itself. If this was the writer's intention, the author would like to heartily concur with him.

Variations in field practice, non-uniformity of materials and labor, uncertainty of vehicular weights and repetitions, etc., all affect the real safety of structures. They must be taken into consideration when choosing load factors. Hence while it is possible to determine a range of values for the load factor applicable to the usual conditions, the integrity, understanding, and experience of the engineer are always vital for the proper design of any structure.

The purpose of the paper was to point out the wide divergence of load factors when designed by empirical values of allowable stresses. It also pointed out that a fixed set of load factors could lead to undesirable behavior under service conditions. Hence a structure is not necessarily properly designed if it meets the requirements of stresses or load factors, or even both. It is the duty of the engineer to design structures that will behave properly under service conditions, and will possess enough overload capacity to meet occasional overloads. The understanding of load factors will help the engineer in his design.

The writer wishes to call the discusser's attention to at least one instance not correctly described in his discussion. The writer mentioned the reinforcement failure in a floor slab when a bar was stressed with inadequate imbedment at an anchorage on a right angle turn of about three foot radius. The writer happens to know of the occurrence and hence wishes to rectify some of the distorted facts in the discussion. First of all, it was not the failure of a bar, but of a cable with 7 strands of wires. When the group of wires were tensioned, they had a tendency to flatten around the curve and hence tended to split the concrete cover. It was not a reinforcement failure, but the spalling of concrete resulting from the splitting action of the wires. Furthermore, the writer cannot see any significance of the Poisson's effect mentioned in the discussion.

The discusser also wrongly attributed the above incident to the Structural Engineers of the Division of Architecture, State of California. Actually, it

a. Proc. Paper 1315, July, 1957, by T. Y. Lin.

1. Prof. of Civ. Eng., Univ. of California, Berkeley, Calif.

was the Contractor's force who misplaced the cable with insufficient cover, thus resulting in the breakage. The breakage of the slab concrete at the bent of the two cables was repaired by the Contractor without cost to the owner. The breakage occurred to only two cables out of several dozens.

It is well known that one of the assets of prestressed concrete was its severe test at transfer. Much of the concrete and steel is subjected to the highest stresses at transfer of prestress. And remedy, if required, can be readily applied before the structure is put into service.

The writer wishes to conclude by saying that the knowledge of the strength of prestressed concrete structures is essential for a good design. But the engineer must also produce a structure that will behave properly under service conditions, i.e., it must not camber, deflect, crack, shrink, creep, or vibrate excessively. Such proper behavior is not measured solely by load factors nor by stresses. Basic understanding is required not only of the theory, but also of the actual field and service conditions. In this regard, the writer agrees with the discusser that experienced analysis and judgment cannot be replaced by some simple rules of thumb.

LATERAL DEFLECTIONS AND STRESSES IN BUILDING FRAMES[a]

Discussion by E. Neil W. Lane

E. NEIL W. LANE,[1] A.M. ASCE.—Any author endeavoring to present his own developments in competition with the highly worked field of stress analysis and deflections, even when limited to a type of load and framing, is to be admired for his efforts and ambitions. In this Mr. McClellan fills the bill.

His theory, as limited by his assumptions, is quite reasonable and fine for the general purpose intended, but it is felt that it would be well to experiment, and study its application, on a wider variety of shape conditions (as to relative spans, story offsets, number of stories and bays, and the like).

On the basis of the author's frame selection from Bulletin 80 of the Illinois Engineering Experiment Station (now, out of print) and applying ten kips of shear per floor, the writer endeavored to do some good with a five inch slide rule in his spare time and came up with the accompanying analysis by the Hardy Cross method to add value to the author's comparison of different means.

At this point, the ease with which the Cross method may be adapted to a change from the academic or unit shears applied in this example to the actual design wind loads used in Bulletin 80 should be emphasized. It is felt as though it lends itself to a better visualized office record of complete and compact nature, as compared to the author's perhaps neater, but more voluminous tabulation on several smaller and certainly more wieldy sheets.

Where the combined stiffness of loading or straining members in a joint (or group) at a floor level is more than four times that of the resisting or restraining members, the story translation becomes dominantly due to joint rotation (80% or more) and the analysis can be further simplified. In the top few stories the column bending portion of the story translation reduces rapidly from three eights of the total for the exterior roof supports down through 27, 24, 18, 16 and 15% at the fifteenth floor. So, in the writer's problem, translation due to column bending is too great a portion of the total (above the fifteenth story) to be overlooked in the design of individual members. Below that, the percentage can be estimated at less than fifteen percent above rotational stresses for reasonable accuracy and less design time.

For the building as a whole the total deflection of 7.25 inches consists of 5.84 inches (80.5%) due to joint rotation and 1.41 inches due to column bending.

The story rate of deflection is also interesting, varying from an average of 0.16 inches for the top five to 0.375 inches from the fifteenth down through the tenth. Then the rate of increase lessens so one finds an average story

a. Proc. Paper 1354, September, 1957, by Robert E. McClellan.

1. Structural Engr., 611 Carlo Court, Lodi, Calif.

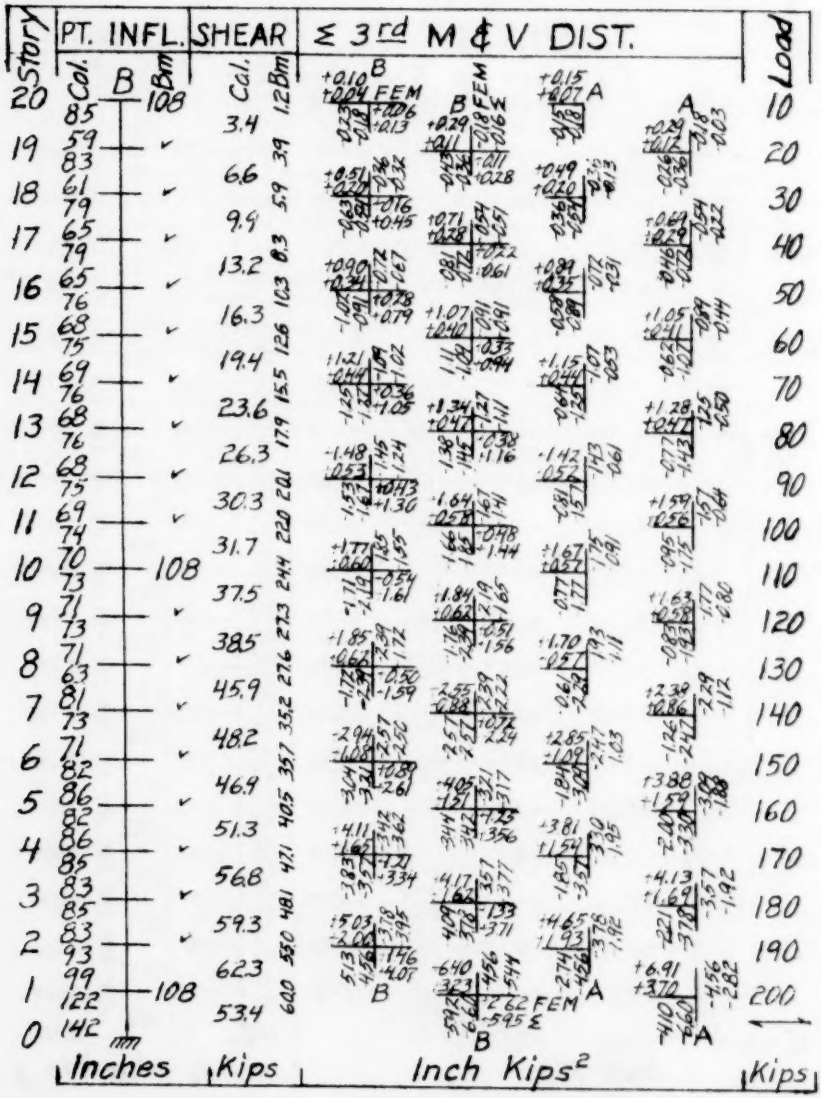
deflection of 0.445 inches between the fifth and tenth floors. The lower five levels average 0.505 inches each. From this it is seen that the lateral deflections from the top tend to increase about parabolically until a reverse curvature sets in ending with a fairly constant slope for the lower half of the frame. This is the primary reason why the author finds closer agreement between the exact deflection of Equation 16 and that calculated by his proposed method, in the lower stories.

For practical construction and use, also because of working joints to be concealed and material's manufacturing tolerances, at ordinary story heights, the maximum deflection has been held around 1/3 + inch in 10 ft.

From the above, it can be deduced that the author's procedure is ideally suited to the fairly uniform portions of tall buildings or parts of moderate change per story, and that framing within, say three stories or bays from the foundations, from the roof, and story offsets, auditoriums, balconies and similar rapid changes in framing should be given additional study. This is believed to be as the author intended. This is also more applicable to the matter of actual deflections in most instances than the usual design stresses at higher allowances where secondary stresses within ten percent are largely omitted.

The effect of the lateral bending of the building frame as a whole on the leeward columns in the first story is to shorten the interior by 0.10 inch and the exterior by 0.20 inch and produce a maximum direct stress of 22,500 psi at the level of the points of contraflexure on the exterior column.

Story	MOMENTS					PT. INFL.		SHEAR		Defl.
	A Col.	Ext. Bm		B Col.	Int. Bm	Col. A	Bm	Col.	13 Bm	
20	-0.19	+0.19	+0.16	-0.29	+0.13	119	-143			0.042
19	-0.64 / -0.33	+1.37	+0.36	-0.20 / -0.54	+0.37	25 / 97	-134	1.6	2.8	0.168
18	-0.16 / -0.45	+1.61	+0.56	-0.40 / -0.74	+0.64	47 / 84	-138	3.4	4.7	0.117
17	-0.28 / -0.59	+1.88	+0.74	-0.64 / -1.04	+0.90	55 / 86	-140	5.1	6.3	0.174
16	-0.46 / -0.71	+1.11	+0.98	-0.86 / -1.24	+1.12	58 / 82	-140	6.9	7.9	0.213
15	-0.54 / -0.82	+1.35	+1.20	-1.11 / -1.46	+1.36	62 / 78	-140	8.7	9.7	0.256
14	-0.70 / -0.92	+1.60	+1.45	-1.34 / -1.80	+1.68	66 / 81	-134	10.6	11.5	0.312
13	-0.72 / -1.11	+1.84	+1.67	-1.60 / -1.99	+1.93	63 / 80	-138	11.4	13.3	0.340
12	-0.85 / -1.20	+2.08	+1.90	-1.79 / -2.27	+2.17	64 / 80	-138	13.8	15.1	0.377
11	-0.95 / -1.35	+2.30	+2.08	-2.09 / -2.36	+2.38	64 / 73	-139	14.9	16.6	0.405
10	-1.30 / -1.24	+2.53	+2.43	-2.21 / -2.74	+2.64	71 / 71	-135	16.4	16.7	0.450
9	-1.29 / -1.33	+2.62	+2.50	-2.62 / -2.81	+2.95	73 / 62	-135	17.5	19.3	0.438
8	-1.77 / -1.01	+2.77	+2.57	-2.25 / -2.80	+2.98	82 / 53	-137	21.5	22.2	0.473
7	-1.75 / -1.73	+3.48	+3.34	-3.61 / -3.53	+3.80	91 / 79	-135	19.1	25.8	0.456
6	-1.41 / -2.32	+3.73	+3.44	-3.43 / -3.87	+3.86	65 / 83	-137	21.8	27.1	0.392
5	-2.39 / -2.44	+4.83	+3.86	-4.04 / -4.20	+4.38	85 / 85	-147	28.1	32.4	0.469
4	-2.38 / -2.33	+4.71	+4.14	-4.42 / -4.81	+5.09	83 / 83	-141	28.7	34.5	0.424
3	-2.41 / -2.75	+5.16	+4.62	-4.74 / -5.09	+5.20	85 / 90	-139	28.2	37.0	0.445
2	-2.40 / -3.10	+5.50	+4.78	-4.91 / -5.80	+5.93	78 / 94	-141	30.7	38.9	0.466
1	-3.19 / -4.48	+7.67	+6.04	-6.15 / -6.48	+6.49	98 / 96	-147	32.7	51.9	0.569
0	-7.82 →	+7.82	+7.62	← -7.62		168		46.6		0.360
	Inch Kips²					Inches		Kips		In.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DESIGN OF MULTI-LEVEL GUYED TOWERS: STRUCTURAL ANALYSIS[a]

Corrections to discussion by Kuang-Han Chu
Discussion by I. M. Nelidov

CORRECTIONS—On page 1576-33, in item 2, the term $^{1}8I_o^2$ should be changed to $'8I_o^2$. In item 5, the term $-\frac{\tau 2}{2} I_o 4(\delta - I_o)^2$ should be changed to $-\frac{\tau 2}{2} I_o^4(\delta - I_o)^2$ and the term $-\frac{\tau 2}{2} I_o 4\delta(\delta - I_o)^2$ should be changed to $-\frac{72}{2} I_o^4 \delta(\delta - I_o)^2$. In item 7, cQ_2^2 should be $-cQ_2^2$. In items 2, 3, 4, and 5 the greek letter delta (δ) should be changed to sigma (σ).

I. M. NELIDOV,[1] M. ASCE.—The authors have presented a very complete study of the subject of multi-level guyed towers with a quite complicated analysis that is well demonstrated in the article. The writer wishes to submit a simplified version of the design which may also be used for a preliminary design.

In this he assumed the following:

1. That the cables are straight without sag;

2. That the horizontal deflections of the cables are predetermined, namely that they are varying proportionally with height;

3. That the base of the tower may be hinged or fixed;

4. That the tower-beam develops full reactions as on unyielding supports, with additional reactions if the base is fixed; the cables supply support for these reactions in accordance with 2 above.

A. Tower with hinged base.

In this case the deflection at any level is $\Delta = \Delta_o \frac{h}{h_o}$, where Δ_o is horizontal deflection at top of tower of height h_o, and Δ and h are respectively deflection and height at any level.

The typical stress diagrams for the stresses in cables at any level are shown on Fig. 1, where R is the reaction of the tower, T_i is the initial stress in cables and T_Δ is the stress in cables due to the deflection Δ.

It follows from Fig. 1 that $T_\Delta = \frac{R}{2\cos\alpha}$ (1)

Tension in the windward cable is $T_w = T_i + T_\Delta$ and that in the leeward cable

a. Proc. Paper 1356, September, 1957, by Edward Cohen and Henri Perrin.
1. Civ. & Structural Engr., San Francisco, Calif.

$T_1 = T_i - T_\sigma$. If it is desired to keep the leeward cable operative, then, by assuming $T_1 = kT$, where $k \leq 1$, we obtain initial tension $T_i = (1 + k)T_\Delta$ and

$$T_w = T_i + T_\Delta = (2 + k)T_\Delta \qquad (2)$$

The relationship between lateral deformation and stress in cable T_Δ can be obtained from Fig. 4*: $\Delta c = \Delta a \cos\alpha$ and $T_\Delta = \frac{\Delta c}{c} EA = \Delta a \cos\alpha \frac{EA}{C}$, (3)

where E is modulus of elasticity of cable and A is the cross-section area of the same.

From exp. (1) and (3), we obtain: $\Delta = \frac{Rc}{2 \cos\alpha \; EA} = \frac{2 \; Th}{Sin 2\theta (EA}$ (4)

The area of the cable A is obtained with the expression: $A = \frac{T_w}{f_t}$ (5)

where f_t is the allowable stress in cable.
Heretofore the stresses and deformations of the cables were considered coinsiding with the plane of action of lateral forces; however with towers where usually either 3 or 4 cables are used, this will not be the case, as shown on Fig. 2.

Therefore, the actual stresses in cables will be $T'_w = \frac{T_w}{2\cos\beta}$ (6)

where β is found from: $\tan\beta = \frac{(B-b)\sin\alpha}{2h}$

Example #1

Given a tower shown on Fig. 3; lateral wind load w = .08k/ft.; loads on cables and their weight are neglected for simplicity; base is hinged.
Bending momets for the tower as a beam on unyielding supports are:

$M_a = M_d = 0$; $M_b = M_c = 320.'$ k; reactions are: $R_a = R_d = 6.4^k$ and $R_b = R_c = 17.6^k$

Stresses on line 1 -1: due to lateral deflections: $T_{d\Delta} = \frac{6.4}{2x.5} = 6.4$ K and $T_{c\Delta} = T_{b\Delta} = \frac{17.6}{2x.5} = 17.6$ k. If we assume k = .25 in exp. (2), then $T_{wd} = 2.25x6.4 = 14.4$ k and $T_{wc} = T_{wb} = 39.6$ k.
The stresses in actual cables will be by exp. (6) as follows.

At D: $\tan\beta = \frac{(353.5-7.5).866}{600} = .500$ and $\cos\beta = .900$ and $T'_{wd} = \frac{14.4}{2x.9} = 8.0$k

At C: $\tan\beta = \frac{346.0x.866}{400} = .750$, $\cos\beta = .800$ and $T'_{wc} = \frac{39.6}{2x.800} = 24.7$k

At B: $\tan\beta = \frac{346.0}{200}$ x.866 = 1.50, $\cos\beta = .555$ and $T'_{wb} = \frac{39.6}{2x.555} = 35.2$ k.

Stresses in actual cables due to lateral deflections are.

$T'_{d\Delta} = \frac{8.0}{2.25} = 3.55$k; $T'_{c\Delta} = \frac{24.7}{2.25} = 10.98$ k; and $T'_{b\Delta} = \frac{35.2}{2.25} = 15.64$ k.

Initial stresses in actual cables will be as follows.

$T'_{di} = 1.25 \times 3.55 = 4.43$ k; $T'_{ci} = 1.25 \times 10.98 = 13.7$ k; and $T'_{bi} = 1.25 \times 15.65 = 19.55$k.

Select cross-sections of cables next. Assume breaking strength of cables $f_u = 200{,}000.$ psi. and with factor of safety 4, design for working stress $f_t = 50{,}000.$ psi.

At D: $T'_{wd} = 8.00$k and $A_d = \frac{8.00}{50.} = .16$ sq.i.

Select Roebling's galvanized bridge strand 1/2" dia. with breaking strength 15.tns and cross-section area A = .149 sq.i.; assume modulus of elasticity of cable $E = 24 \times 10^3$ ksi. The deflection at D is then: $\Delta_d = \frac{2 \times 3.55 \times 600 \times 12}{.866 \times 24 \times 10^3 .149} = 16.5" = 1.37' = \frac{H_d}{438}$

The required deflections are, at C: $\Delta_c = 16.5 \frac{400}{600} = 11"$; at B: $\Delta_b = 16.5 \frac{200}{600} = 5.5"$

The cross-section area at C: $A_c = \frac{2 \times 10.98 \times 400 \times 12}{.866 \times 24 \times 10^3 \times 11} = .456$ sq.i.

Select 7/8" cable with breaking strength of 46.tns. and an area A = .467 sq.i.

The actual deflection at C will be: $\Delta_c = 11 \frac{.456}{.467} = 10.75"$.

The cross-section at B is: $A_b = \frac{2 \times 15.64 \times 200 \times 12}{.866 \times 24 \times 10^3 \times 5.5} = .655$ sq.i.

Select 1 1/16" cable with breaking strength of 69.tns. and area .663 sq.i.

The actual deflection at B will be: $\Delta_b = 5.5 \frac{.655}{.663} = 5.43"$.

The discrepancies in actual deflections from theoretical are so small that the moments induced by them and reactions will not affect the sizes of the cables.

Example #2.

Given a tower as in example #1, except that the base is fixed.

The moments in the tower acting as a continuous beam on unyielding supports are obtained by one of the well known methods, as follows: $M_a = -277.'$k; $M_b = \pm 247.'$k; $M_c = \pm 339.'$k. and $M_d = 0$. The resulting reactions are: $R_a = 8.15$ k., $R_b = 15.4$ k.; $R_c = 18.15$k and $R_d = 6.3$ k.

The stress due to deflection on line 1-1 is: $T_{d\Delta} = \frac{6.3}{2x.5} = 6.3$ k. and total stress is: $T_{wd} = 2.25 \times 6.3 = 14.2$ k.

Stresses in actual cable are: total stress $T'_{wd} = \frac{14.2}{2x.9} = 7.9$ k. and stress due to deflection is $T'_{d\Delta} = \frac{7.9}{2.25} = 3.5$ k.

The area of cable at D is: $A_d = \frac{7.9}{50.} = .158$ sq.i.

Select 1/2" cable with area .L49 sq.i.; then deflection at D will be:

$$\Delta_d = \frac{2 \times 3.5 \times 600 \times 12}{.866 \times 24 \times 10^3 \times .149} = 16.3" = 1.36' = \frac{H_d}{441}$$

The deflections at points C and B are computed on the basis that the traverse of the deflected tower will be vertical to half of the first tier from where the deflections will be proportional to height, or: $\Delta_c = 16.3 \frac{400-100}{600-100} = 8.78"$ and $\Delta_b = 16.3 \frac{200-100}{500} = 3.26"$. Assuming that the legs of the tower are of angles 4 x 4 x 1/4-1 ea. leg, and that they are on 15. ft. back to back, we obtain the moment of inertia one frame, disregarding the web members for simplicity: $I = 1.94 \times 2 \times 2 \times 90^2 = 63{,}000."^4$

With this, the fixed-end bending moment at A and B will be:

$$-6 - \frac{24 \times 10^3 \times 63{,}000 \times 3.26}{2400^2 \times 12} = -42.7'k$$

Moments at C and D are zero, because of the proportionality of deflections. The moments in the tower beam due to these deflections are: $M_a = -31.'k$; $M_b = \mp 19.8'k$; $M_c = \pm 4.9'k$. and $M_d = 0$. The respective reactions are: $R_a = .26$ k; $R_b = -.38$k; $R_c = .14$k; and $R_d = -.02$ k. Minus sign signifies counter-clockwise moment and a downward reaction. Total reactions for unyielding and deflected supports are: $R_a = 8.4$ k; $R_b = 15.$ k; $R_c = 18.3$ k; and $R_d = 6.3$ k.

Stresses on line 1-1 due to these deflections are: $T_{d\Delta} = \frac{6.3}{2 \times .5} = 6.3$ k; $T_{c\Delta} = \frac{18.3}{2 \times .5} = 18.3$ k; $T_{b\Delta} = \frac{15.}{2 \times .5} = 15.$ k. Total stresses are: $T_{wd} = 2.25 \times 6.3 = 14.2$ k; $T_{wc} = 2.25 \times 18.3 = 41.2$ k; and $T_{wb} = 2.25 \times 15. = 33.8$ k.

Stresses in actual cables are: $T'_{wd} = \frac{14.2}{2 \times .9} = 7.9$ k; $T'_{wc} = \frac{41.2}{2 \times .800} = 25.7$ k; and $T'_{wb} = \frac{33.8}{2 \times .555} = 30.1$ k. Stresses in actual cables due to deflections are: $T'_{d\Delta} = \frac{7.9}{2.25} = 3.5$ k; $T'_{c\Delta} = \frac{25.7}{2.25} = 11.4$ k; and $T'_{b\Delta} = \frac{30.1}{2.25} = 13.4$ k.

Initial stresses in actual cables are: $T'_{di} = 1.25 \times 3.5 = 13.5$ k; $T'_{ci} = 1.25 \times 11.4 = 14.2$ k; and $T'_{bi} = 1.25 \times 13.4 = 16.7$ k.

The cross-section areas of the cables are as follows.

At D: $A_d = .158$ sq.i. as before; 1/2" cable as previously selected; and deflection $\Delta_d = 16.3"$, as before.

At C, with deflection $\Delta_c = 8.78"$, $A_c = \frac{2 \times 11.4 \times 400 \times 12}{.866 \times 24 \times 10^3 \times 8.78} = .593"$

Select 1" cable with breaking strength of 61.tns and an area A = .577 sq.i.

The corrected deflection then will be; $\Delta_c = 8.78 \frac{.593}{.577} = 9.1"$

At B : $A_b = \frac{2 \times 13.4 \times 200 \times 12}{.866 \times 24 \times 10^3 \times 3.26} = .947$ sq.i.

Select 1 1/4" cable with breaking strength of 96.tns. and an area A = .931 sq.i.

The corrected deflection will be: $\Delta_b = 3.26 \frac{.947}{.931} = 3.31"$.

The fixed-end moments resulting from these deflections are:

$M_a = \frac{3.31}{3.26} 42.7 = 43.5'\ k_c M_b$; $M'_b = 42.7 \frac{9.81 - 9.10}{3.26} = 9.3'$ k.

If desired, the recomputation of the moments in the tower beam could be carried with these fixed-end moments, but insofar as it will be just a repetition of the previous step, the computations will be stopped here.

The vertical pressures in the tower can be computed from initial stresses in the cables listed above with the aid of the diagram in Fig. 1a.

Final moments in the twoer as a continuous beam are: $M_a = -277. - 31. = -308.'$k. $M_b = 247. - 19.8 = 227.2'$k $= -M'_b$; $M_c = 339. + 4.9 = 343.9 = -M'_c$; $M_d = 0$. From these moments and the loads w shears can be easily computed. The tower trusses are then designed for these moments and shears. The cable anchors are designed for the maximum cable pulls with a factor of safety of not less than 1.5.

Cables endowed with weight, and sag.

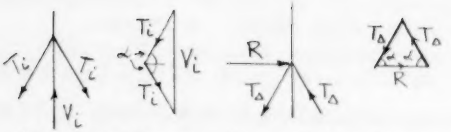
The computations in this case are much more numerous even in the case of assumption that the deflections are to be linear, and will not be presented here.

First, the diagram in Fig. 1c has to be adjusted to new angles θ_1 and θ'_1 between the cables and the tower; this can be done by using the previously computed stresses in the cables by a simplified method and using equations 8, 9 and 17 of the authors.

The lateral deformations can be determined with authors' eq. 14.

The ratio between the total stress in the leeward cable and stress due to deflection could be assumed and cross-sections of cable determined.

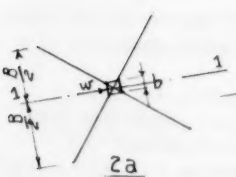
The article presented by the authors should be considered as a classic contribution to the subject of tall towers supported by guy wires.



1a - Initial Stresses 1b - Deflection Stresses

1c - total stresses

Fig. 1



2a

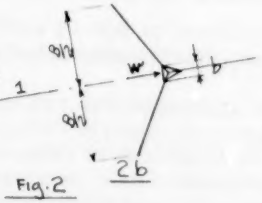


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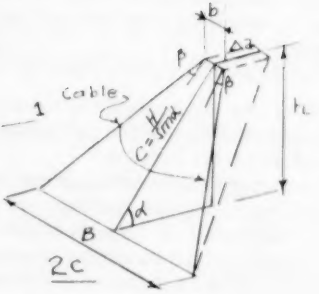
Fig. 2



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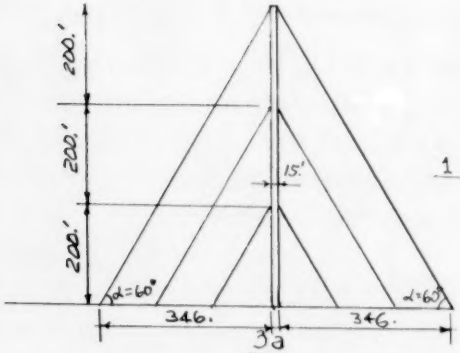
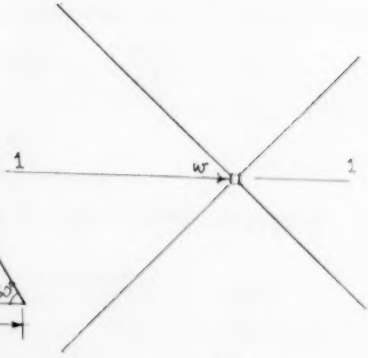


3a



3b

Fig. 3

WOOD DIAPHRAGMS: REPORT OF A SUB-COMMITTEE OF THE COMMITTEE ON TIMBER STRUCTURES[a]

Discussion by John Steinbrugge

JOHN STEINBRUGGE,[1] J.M. ASCE.--The sub-committee's progress report on Wood Diaphragms is a welcome addition to the profession's store of knowledge on diaphragms.

The writer would like to bring to the sub-committee's attention many more tests on materials which should be included under the classification of Wood Diaphragms; principally these tests were performed on various insulating boards which are composed of cemented wood fibers or somewhat similar related materials. The various tests are attached in the form of a bibliography with this discussion.

In general the insulating-board tests can be differentiated in the manner of attachment of the sheets to the supports; these attachments were:

a. Common or box nails
b. Nails with large washers or a continuous metal nailing strip.
c. Glue with some combination of the above nailing.

As could be expected the diaphragms that only utilized nails gave the lower ultimate loads and greatest amount of flexibility; in general, building officials have allowed from 35 to 100 pounds per lineal foot shear for these materials. Various tests have been performed attempting to correlate thickness, type of material, and size of nail with the ultimate load and deflection capacity of the diaphragm. This has proven impossible due to the great variation in the materials. In other words the use of larger nails for a particular material may not result in a stiffer diaphragm or even a higher ultimate load. The use of washers or a continuous metal nailing strip through which the nails are driven have proven to give considerable increase in strength and stiffness in some cases. Every material should be investigated to determine the method of attachment which produces the best possible results. The load-deflection curves for about half of the tests have proven to be stiffer than the conventional diagonally sheathed diaphragms. However, the ultimate loads of the nailed insulating board diaphragms are usually less than diagonal sheathing.

The glued insulating board diaphragms have proven to be very stiff and to have high ultimate loads; this type of diaphragm has been allowed from 150 to 370 pounds per lineal foot shear value by various building officials. The load-deflection curve for this type of diaphragm is about the same as for the blocked plywood diaphragms.

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a. Proc. Paper 1433, November, 1957, by a sub-committee of the committee on Timber Structures of the Structural Division.

1. Structural Engr., C. F. Braun & Co., Alhambra, Calif.

The report notes that insufficient data was available concerning the action of wallboard sheathed, and plaster sheathed diaphragms. Recently a number of excellent tests on these materials have come to the writer's attention, and they are noted in the attached bibliography. The tests were performed in accordance with ASTM E-72, and the various code officials have assigned them permissible shear values ranging from 120 to 275 plf for various gypsum wallboard attachments and 215 to 265 plf for various gypsum plaster attachments. Many different attachments of the materials were tested so as to provide a wide range of stiffness and strength.

The writer has also included in the attached bibliography other tests that concern diagonally and straight sheathed diaphragms and special laminated wood panels.

It would be highly desirable if the sub-committee could include in their report recommended factors of safety and permissible deflections. Many of the allowable shears that are presently permitted for the materials in the report have very different factors of safety and actual deflections.

The biggest problem that still confronts the profession in regards to the performance of diaphragm materials during seismic activity is the lack of dynamic data on their action. In other words will the relatively weak and porous insulating boards show increasing deflections after many reversals of load as experienced in an earthquake, or will the application of a single seismic movement be of such short duration that this is of no importance? For vertical loading, wood members are permitted a 100% increase in stress due to impact loading; shouldn't this same factor be used when the materials are subject to earthquake loads instead of only 33%? The static tests presently being performed on the diaphragm materials should give an adequate indication of the strength and stiffness for wind loads, but there can be significant errors for seismic loading.

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Test No. 7118J, 1952
Test No. M1038-0, 1953 and 1954
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15. Racking Load Tests by Armour Research Foundation.

Test No. 6117D, Jan 3, 1951, Diagonal and horizontal sheathed.
Test No. K1037, April 25, 1956, Diagonal sheathed.
Test No. K1057, June 11, 1956, Diagonal sheathed.

LAMINATED WOOD PANELS

16. "Structural Tests of Lincoln Sandwich Wall Panels" by R. E. Davis, April 24, 1956.

17. "Tests of Laminated Wood Panels, (Forestwall Panels)," by R. E. Davis, May 17, 1956.

18. Racking Load Test on Pre-Fabricated Panel Assembly, Cliff May-Chris Choate, by Triplett and Barton Testing Laboratory.

THE DESIGN OF RIGID FRAME BENTS[a]

Discussion by F. J. Tamanini, Howard H. Mullins, R. E. Bowles, Robert B. Harris and Albert D. M. Lewis

F. J. TAMANINI,[1] M. ASCE.—The author is quite correct in his statement that the engineer must overcome the dilemma of which tool to select in the solution of the various types of structures which he is apt to encounter. He has presented an ideal structural frame to illustrate solution by three universally accepted methods: Moment Distribution, Column Analogy, and Slope Deflection. In presenting the methods, Mr. Zimmermann has not taken advantage of known short cuts or abbreviated schemes which writers present from time to time in various publications. This was no doubt intentional since a comparison of the basic methods of analyses was intended only and since an exceedingly large number of such techniques are available. Many engineering offices have such data available in the form of charts, nomographs, and tables compiled after years of practice to simplify analysis and thereby speeding up structural analysis so that the engineer can devote more time to design work.

In discussing the moment distribution solution, the author made mention of the disadvantage of "assuming "Δ" of some fraction of an inch and then calculate the corresponding column fixed end moments directly." The writer agrees with the conclusion of this statement but submits for consideration the thought that it is not necessary to assign a value to "Δ." The following approach (Figs. 1 and 2) is a direct solution of the basic bent by two applications of moment distribution; namely, transverse loads and displacement (or translation).

Analysis of a structure by slope deflection is a well-known but tedious operation causing many to become disheartened by the apparent accuracy of procedure required. The writer presents the following solution (Fig. 3) of the basic problem by Amirikian's Abbreviated Slope Deflection Equation presented in the author's first reference in his presentation. The simplicity and directness of solution by this method are obvious and is but a typical application of this method.

The uniqueness of Mr. Amirikian's rendition of the slope deflection equation is brought about by the fact that one equation is written for each "joint" of a structure (simple and fixed supports excepted) and one shear equation is written for each story or bay of a structure where translation occurs. This is done in lieu of conventionally writing a slope deflection equation for each end of a member in a framed structure. It is based upon the fact that the summation of moments at a joint is equal to zero. Further, simplicity is manifested by the following substitutions for the well-known values of angular

a. Proc. Paper 1434, November, 1957, by Richard Z. Zimmermann.

1. Engr. Research and Development Labs., Fort Belvoir, Va.

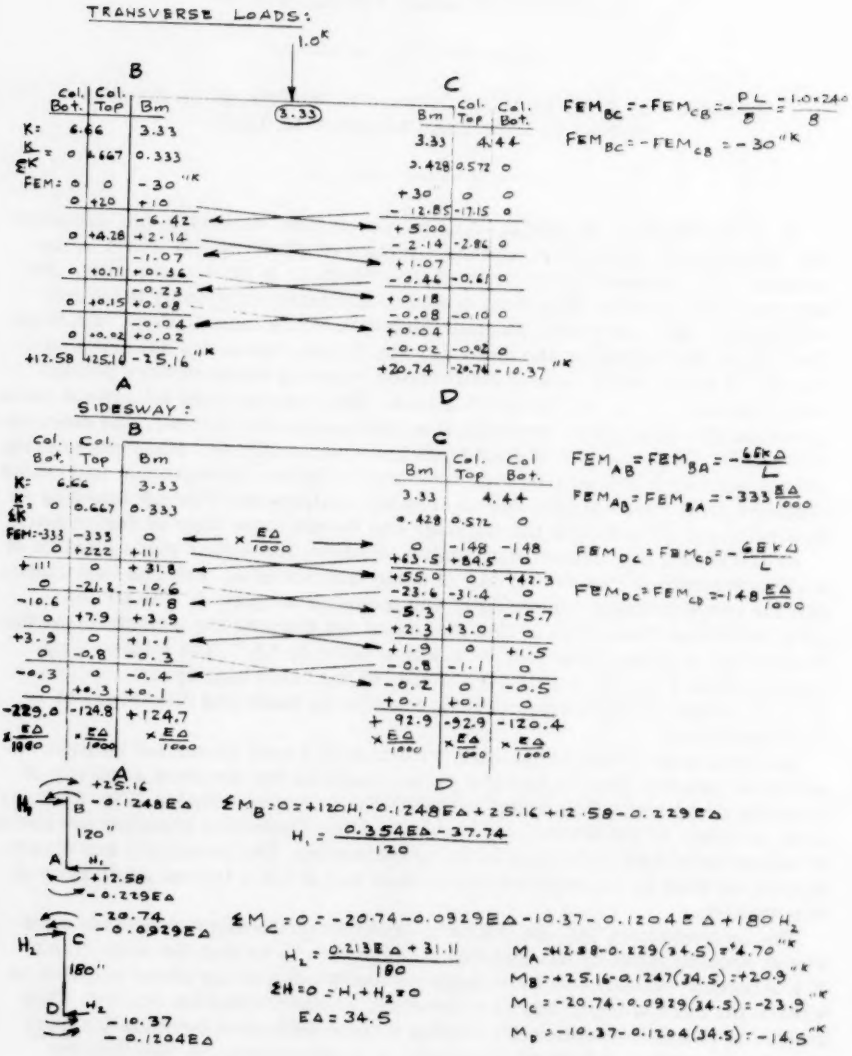

TRANSVERSE LOADS:
1.0K
3.33
B
C
A
D
Col. Bot.
Col. Top
Bm
K:
ΣK
FEM:
+12.58
-25.16
-10.37 "K
SIDESWAY:
FEM_AB = FEM_BA = -6EKΔ/L
FEM_AB = FEM_BA = -333 EΔ/1000
FEM_DC = FEM_CD = -6EKΔ/L
FEM_DC = FEM_CD = -148 EΔ/1000
ΣM_B = 0 = +120H1 - 0.1248EΔ + 25.16 + 12.58 - 0.229EΔ
H1 = (0.354EΔ - 37.74)/120
ΣM_C = 0 = -20.74 - 0.0929EΔ - 10.37 - 0.1204EΔ + 180H2
H2 = (0.213EΔ + 31.11)/180
ΣH = 0 = H1 + H2 = 0
EΔ = 34.5
M_A = +12.58 - 0.229(34.5) = +4.70 "K
M_B = +25.16 - 0.1247(34.5) = +20.9 "K
M_C = -20.74 - 0.0929(34.5) = -23.9 "K
M_D = -10.37 - 0.1204(34.5) = -14.5 "K

Fig. 1.

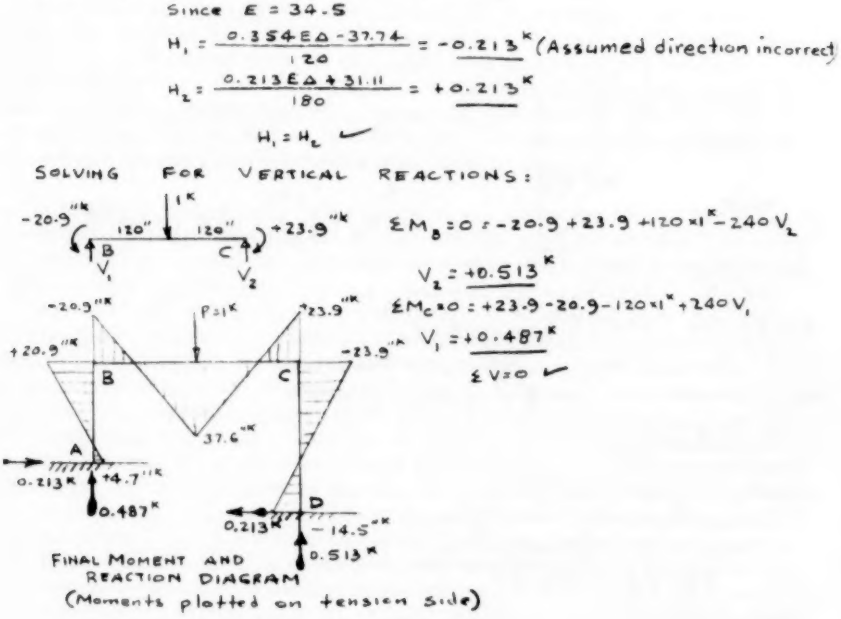


Fig. 2.

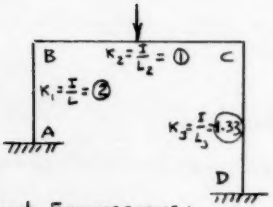
rotations and joint translations (members having constant EI values):

$$A = 4E\theta_A,\ B = 4E\theta_B,\ C = 4E\theta_C, \text{ etc.}$$

$$R_{AB} = \frac{4E\Delta}{L_{AB}},\ R_{CD} = \frac{4E\Delta}{L_{CD}}, \text{ etc.}$$

Proceeding now with the direct solution of the basic rigid frame by using Abbreviated Slope Deflection Equations, we write the required "joint equation" at B and at C together with the third "shear equation" for the members AB and CD. For the purpose of demonstration the shear expression (Equation 3) is developed. This need not be done for it can be written directly when one becomes more familiar with this system of solution.

The values obtained by each of the above direct methods check each other very closely as well as those of the author. It is obvious that the reaction values obtained from the latter method would be almost identical to those of the former method. No mention is made of the classical method of analysis,--the method of least work. Where the designer is confronted with changing condition of loading or frequent analysis of basically the same type of rigid frame bent, analysis by work or energy equations would be a most profitable choice for the development of influence lines or tables. The generation of table influence lines by digital computers using least work could be most



$$FM_{BC} = -\frac{PL}{8} = \frac{1.0(20)}{8} = -2.5\ ^{'K}$$

$$FM_{CB} = +2.5\ ^{'K}$$

$$R_1 L_1 = R_3 L_3$$

$$R_3 = \frac{L_1}{L_3} R_1 = \frac{10}{15} R_1 = 0.667 R_1$$

Joint Expressions:

A+ B, $\quad B(K_1+K_2) + \frac{K_2 C}{2} - K_1 R_1 + FM_{BC} = 0$

Eq. 1: $\quad B(3) + \frac{C}{2} - 2R_1 - 2.5 = 0$

A+ C, $\quad C(K_2+K_3) + \frac{K_2 B}{2} - K_3 R_3 + FM_{CB} = 0$

Eq. 2: $\quad C(2.33) + \frac{B}{2} - 1.33(0.667 R_1) + 2.5 = 0$

Shear Expression:

In member AB:

$$\Sigma M_B = 0 = M_{AB} + M_{BA} + 10 H_A$$

$$M_{AB} = \cancel{K_1 A}^{\,0} + \frac{K_1 B}{2} - K_1 R_1 + \cancel{FM_{AB}}^{\,0}$$

$$M_{BA} = K_1 B + \cancel{\frac{K_1 A}{2}}^{\,0} - K_1 R_1 + \cancel{FM_{BA}}^{\,0}$$

$$3B - 4R_1 + 10 H_A = 0$$

In member DC:

$$\Sigma M_C = 0 = M_{CD} + M_{DC} + 15 H_D$$

$$M_{CD} = K_3 C + \cancel{\frac{K_3 D}{2}}^{\,0} - K_3 R_3 + \cancel{FM_{CD}}^{\,0}$$

$$M_{DC} = \cancel{K_3 D}^{\,0} + \frac{K_3 C}{2} - K_3 R_3 + \cancel{FM_{DC}}^{\,0}$$

$$2.0 C - 2(1.33) 0.667 R_1 + 15 H_D = 0$$

Since $H_A + H_D = 0$

Eq. 3: $\quad 4.5 B + 2.0 C - 7.7 R_1 = 0$

From Equations 1, 2, and 3:

$$R_1 = 0.477 \ ; R_3 = 0.318$$

$$C = -1.18$$

$$B = +1.35$$

$$M_{AB} = K_1 \frac{B}{2} - K_1 R_1 = 2(0.675 - 0.477)$$

$$M_{AB} = +0.396\ ^{'K} = +4.75\ ^{"K}$$

$$M_{BA} = K_1 B - K_1 R_1 = 2(1.35 - 0.477)$$

$$M_{BA} = +1.75\ ^{'K} = +21.0\ ^{"K}$$

$$M_{BC} = K_2 B + \frac{K_2 C}{2} + FM_{BC} = 1.0(1.35 - 0.59) - 2.5$$

$$M_{BC} = -1.74\ ^{'K} = -20.9\ ^{"K}$$

$$M_{CB} = K_2 C + \frac{K_2 B}{2} + FM_{CB}$$

$$M_{CB} = 1.0(-1.18 + 0.675) + 2.5$$

$$M_{CB} = +1.99\ ^{'K} = +23.9\ ^{"K}$$

$$M_{CD} = K_3 C - K_3 R_3 = 1.33(-1.18 - 0.318)$$

$$M_{CD} = -1.99\ ^{'K} = -23.9\ ^{"K}$$

$$M_{DC} = K_3 \frac{C}{2} - K_3 R_3 = 1.33(-0.59 - 0.318)$$

$$M_{DC} = -1.21\ ^{'K} = -14.6\ ^{"K}$$

Fig. 3.

worthwhile. For single applications, the method of least work is not believed to be as economical of time or effort as the methods indicated by the author.

In conclusion, it must be borne in mind that in selecting a particular method of analysis for a structure, one must not only take advantage of the simplicity of the basic methods of analyses but must also use whatever adaptations of them that are available. The author has certainly brought to light a very stimulating subject by his paper, one which is not important because of any new postulations, but because of the generated discussions on the subject among those concerned with structural design.

HOWARD H. MULLINS.[1]—The author is to be commended for his effort in comparing, for the benefit of the profession, the work involved in computing the stresses in a three-fold statically indeterminate bent by three separate methods now in general use.

Any one method of analysis cannot generally be applied with equal facility to all types of structures under all loading conditions. Any one method of analysis may, indeed, be unsuitable for the same structure under different loading conditions. The structure selected by the author, for comparing three methods of analysis, is typical of the type which all State Bridge Departments are frequently called upon to design as highway or railway grade-separation structures except that such structures generally have varying sections and, therefore, variable moment of inertia. In designing such structures, however, the designer is not interested only in the effect of a concentrated load at the center of the frame but also in the effect of moving loads over the beam of the frame and the effect of earth-pressure against the legs. In such a case one would try to select a method of analysis which will permit the computation of influence lines from which the critical conditions of loading may be determined. It does not appear that either of the three methods selected by the author for comparison is the most suitable for this purpose. The writer is of the opinion that a straight forward attack by one of the socalled classical methods of analysis—such as the method of least work for example—will yield a solution about as expeditiously as any of the "special case methods."

In the analysis of such structures one can reduce the work of computation greatly by a proper selection of the redundants, or by shifting the axes of coordinates, and by using the method of elastic weights for computing the deflection of the statically determinate system.

Because the structure used by the author, as an example, is three-fold indeterminate, three elastic equations are required in order to compute the stresses. In such a structure as this the work done in resisting the direct stresses is generally quite small in comparison with the work done in resisting the bending stresses and may usually be neglected with neglible error. The work performed by such a structure in supporting externally applied loads is then,

$$W = \tfrac{1}{2} \sum \frac{M^2 \Delta S}{EI} \qquad (1)$$

in which,

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M = moment at any point in the structure due to the applied loads,

ΔS = finite length over which **M** is assumed to be constant,

E = modulus of elasticity of the material, and

I = moment of inertia of the ΔS section.

If we let X_a, X_b and X_c represent the three statically indeterminate parameters we may write,

$$M = M_o + M_a X_a + M_b X_b + M_c X_c \tag{2}$$

Then,

$$w = \frac{1}{2}\sum \frac{(M_o + M_a X_a + M_b X_b + M_c X_c)^2 \Delta S}{EI} \tag{3}$$

in which,

M_o = the moment in the statically determinate system due to the external loads only,

M_a = the moment in the statically determinate system due to $X_a = 1$,

M_b = the moment in the statically determinate system due to $X_b = 1$, and

M_c = the moment in the statically determinate system due to $X_c = 1$.

If we differentiate Equation (3) with respect to each of the unknowns X_a, X_b and X_c, the three elastic equations are,

$$\sum \frac{M_o M_a \Delta s}{EI} + X_a \sum \frac{M_a^2 \Delta s}{EI} + X_b \sum \frac{M_a M_b \Delta s}{EI} + X_c \sum \frac{M_a M_c \Delta s}{EI} = 0 \tag{4}$$

$$\sum \frac{M_o M_b \Delta s}{EI} + X_a \sum \frac{M_a M_b \Delta s}{EI} + X_b \sum \frac{M_b^2 \Delta s}{EI} + X_c \sum \frac{M_b M_c \Delta s}{EI} = 0 \tag{5}$$

$$\sum \frac{M_o M_c \Delta s}{EI} + X_a \sum \frac{M_a M_c \Delta s}{EI} + X_b \sum \frac{M_b M_c \Delta s}{EI} + X_c \sum \frac{M_c^2 \Delta s}{EI} = 0 \tag{6}$$

It will not be necessary to solve these three simultaneous equations if we select the axes of coordinates so that,

$$\left.\begin{array}{ll} \text{(a)} & \sum \frac{M_a M_b \Delta s}{EI} = 0 \\ \text{(b)} & \sum \frac{M_a M_c \Delta s}{EI} = 0 \\ \text{(c)} & \sum \frac{M_b M_c \Delta s}{EI} = 0 \end{array}\right\} \tag{7}$$

For the structure shown in Fig. 1,

$$\left.\begin{array}{ll} \text{(a)} & M_a = y \\ \text{(b)} & M_b = x \\ \text{(c)} & M_c = 1 \end{array}\right\} \quad (8)$$

and Equations 7 become,

$$\left.\begin{array}{ll} \text{(a)} & \sum \frac{xy\Delta s}{I} = 0 \\ \text{(b)} & \sum \frac{y\Delta s}{I} = 0 \\ \text{(c)} & \sum \frac{x\Delta s}{I} = 0 \end{array}\right\} \quad (9)$$

Equations 4, 5 and 6 now become,

$$\sum \frac{M_o y \Delta s}{I} + X_a \sum \frac{y^2 \Delta s}{I} = 0 \quad (10)$$

$$\sum \frac{M_o x \Delta s}{I} + X_b \sum \frac{x^2 \Delta s}{I} = 0 \quad (11)$$

$$\sum \frac{M_o \Delta s}{I} + X_c \sum \frac{\Delta s}{I} = 0 \quad (12)$$

from which,

$$X_a = - \frac{\sum \frac{M_o y \Delta s}{I}}{\sum \frac{y^2 \Delta s}{I}} \quad (13)$$

$$X_b = - \frac{\sum \frac{M_o x \Delta s}{I}}{\sum \frac{x^2 \Delta s}{I}} \quad (14)$$

$$X_c = - \frac{\sum \frac{M_o \Delta s}{I}}{\sum \frac{\Delta s}{I}} \quad (15)$$

Referring to Fig. 1 one notes that

$$x_o = \frac{\sum \frac{x_1 \Delta s}{I}}{\sum \frac{\Delta s}{I}} \quad (16)$$

and

$$y_o = \frac{\sum \frac{y'\Delta s}{I}}{\sum \frac{\Delta s}{I}} \tag{17}$$

For an unsymmetrical structure, it is necessary to rotate the X-axis so that $\sum \frac{xy\Delta s}{I} = 0$

We write,

$$y = \bar{y} \pm mx \tag{18}$$

Then

$$\sum \frac{x\bar{y}\Delta s}{I} \pm m \sum \frac{x^2 \Delta s}{I} = 0 \tag{19}$$

from which,

$$m = \pm \frac{\sum \frac{x\,\bar{y}\Delta s}{I}}{\sum \frac{x^2\Delta s}{I}} \tag{20}$$

The summation for the numerators of Equations (13), (14) and (15) extends only over that part of the structure stressed by the external loads. The summation for the denominators extends over the whole structure.

Because the geometry of the structure treated by the author consists of straight lines only and the moment of inertia is constant throughout, one may replace the summation signs in the previous equations, by that of integration. The elastic center is located by Equations (16) and (17). Thus,

$$\begin{array}{l} 10 \times 0 = 0 \\ 20 \times 10 = 200 \\ \underline{15 \times 20 = 300} \\ 45 \end{array} \qquad x_o = \frac{500}{45} = 11.11'$$

$$\begin{array}{l} 10 \times 5 = 50 \\ 20 \times 10 = 200 \\ \underline{15 \times 2.5 = 37.5} \\ 45 \end{array} \qquad y_o = \frac{287.5}{45} = 6.39'-$$

We now evaluate $\int x^2 ds$ for use in Equations (14) and (20),

$$\sum \int x^2 ds = \left[\frac{x^3}{3}\right] + 10x^2 + 15x^2$$

$$= \begin{cases} \frac{1}{3}(\overline{11.11}^3 + \overline{8.89}^3) = 691 \\ 10 \times \overline{11.11}^2 = 1234 \\ 15 \times \overline{8.89}^2 = 1185 \\ \sum \int x^2 ds = 3110 \end{cases}$$

and $\int x\bar{y} ds$ for use in Equation (20).

$$\sum\int x\bar{y}ds = 3.61\left(\int_o^{8.89} xdx - \int_o^{11.11} xdx\right) - 11.11\int ydy + 8.89\int ydy$$

$$= \begin{cases} \frac{3.61}{2}\left(-\overline{11.11}^2 + \overline{8.89}^2\right) & = -80.1 \\ \frac{11.11}{2}\left(-\overline{3.61}^2 + \overline{6.39}^2\right) & = 154.4 \\ \frac{8.89}{2}\left(\overline{3.61}^2 - \overline{11.39}^2\right) & = -518.5 \\ \sum\int x\bar{y}ds & = -444.2 \end{cases}$$

The rotation of the X-axis is, therefore,

$$m = \frac{444.2}{3110} = 0.1428$$

Using this value of m = tan θ, one finds for the limits of y the dimensions shown on Fig. 2. Starting at the left leg one may write, for the beam,

$$y = 2.02 + mx.$$

Then,

$$\int y^2dy = \sum\int y_L^2dy + \int_o^{20}\left(2.02 + mx\right)^2dx$$

$$= \begin{cases} \overline{2.02}^2 \times 20 & = 81.6 \\ \frac{4.04}{2} \times 0.1428 \times \overline{20}^2 & = 115.4 \\ 0.1428^2 \times \frac{\overline{20}^3}{3} & = 54.4 \\ \frac{1}{3}\left(\overline{2.02}^3 + \overline{7.97}^3 + \overline{4.88}^3 + \overline{10.12}^3\right) & = 555.4 \\ \sum\int y^2dy & = 806.8 \end{cases}$$

If one is interested only in the effect of a single load at some position on the frame, the numerators of Equations (13), (14) and (15) are evaluated. For the case treated by the author one finds, for a unit load at the center line of the frame,

$M_o = x - 1.11$ and for the beam,

$y = 3.61 - mx.$

Then

$$\int M_o y ds = \int_{1.11}^{11.11} (x - 1.11)\ (3.61 - mx)\ dx + 10 \sum \int y dy$$

or

$$= \begin{cases} \frac{3.61}{2}\left(\overline{11.11}^2 - \overline{1.11}^2 \right) = 220.5 \\ -4.01\left(11.11 - 1.11 \right) = -40.1 \\ -\frac{0.1428}{3}\left(\overline{11.11}^3 - \overline{1.11}^3 \right) = -65.2 \\ \frac{1.11 \times 0.1428}{2}\left(\overline{11.11}^2 - \overline{1.11}^2 \right) = 9.7 \\ 10 \times \frac{\overline{2.02}^2}{2} = 20.4 \\ -10 \times \frac{\overline{7.98}^2}{2} = -318.0 \\ \hline \int M_o y ds = -172.7 \end{cases}$$

and

$$X_a = \frac{172.7}{806.8} = 0.214$$

$$\int M_o x ds = \int_{1.11}^{11.11} (x - 1.11)\, x dx + 10^2 \times 11.11$$

$$= \begin{cases} \frac{1}{3}\left(\overline{11.11}^3 - \overline{1.11}^3 \right) = 457 \\ -\frac{1.11}{2}\left(\overline{11.11}^2 - \overline{1.11}^2 \right) = -68 \\ 10^2 \times 11.11 = 1111 \\ \hline \int M_o x ds = 1500 \end{cases}$$

and

$$X_b = -\frac{1500}{3110} = -0.4823$$

$$\int M_o ds = \int_{1.11}^{11.11} (x - 1.11)\, dx + 10^2$$

$$= \begin{cases} \frac{1}{2}\ 11.11^2 - 1.11^2 = 61.1 \\ -\ 1.11\ \ 11.11 - 1.11 = -\ 11.1 \\ 10^2 = \underline{100.0} \end{cases}$$

$$\int M_0 ds = 150$$

and

$$X_c = -\frac{150}{45} = -\ 3.333'\#$$

Using these influence values of the redundants X_a, X_b and X_c and placing a 1000# load at the center of the frame, the moments are as follows:

at A, 3333 + 482 x 11.11 + 214 x 7.98 - 1000 x 10 = 396'#
B, 3333 + 482 x 11.11 - 214 x 2.02 - 1000 x 10 = -1744'#
, 3333 + 482 x 1.11 - 214 x 3.45 = 3130'#
C, 3333 - 482 x 8.89 - 214 x 4.88 = -2006'#
D, 3333 - 482 x 8.89 + 214 x 10.12 = 1214'#

In the case where it is desirable to compute the values of the redundants for many different positions of the load it is preferable to use a different approach to that of evaluating the numerators of Equations (13), (14) and (15). Perhaps the best method is to load the two statically determinate cantilevers as shown in Fig. 2 with the $\frac{M}{EI}$ diagram due to $X_a = 1$, $X_b = 1$ and $X_c = 1$. These elastic load diagrams are shown in Fig. 4(a), Fig. 5(a) and Fig. 6(a). The deflection of any point of these statically determinate cantilevers, under the action of the $\frac{M}{EI}$ diagram due to $X_a = 1$, $X_b = 1$ and $X_c = 1$, divided by the denominator of Equations (13), (14) or (15) will give the influence ordinate of the redundant for a load at that point on the frame. The deflection of the statically determinate cantilevers can best be determined by considering the leg as a free cantilever fixed at B or C. Calculate the moments at the desired points in this cantilever under the ($\frac{M}{EI}$ - diagram). Next, consider the beam of the frame fixed on a vertical line through the elastic center. Apply the area of the $\frac{M}{EI}$ = diagram for the leg as a concentrated load at the end of this cantilever. Load the cantilever with its own $\frac{M}{EI}$ - diagram and compute the moments at the desired points. A typical diagram illustrating this procedure is shown in Fig. 3. The bending moments calculated in this manner are shown in Fig. 4(b), Fig. 5(b) and Fig. 6(b).

With the values of the redundants known an influence line may be computed for any point desired.

This method of analysis is perhaps most advantageous in analyzing frames of variable section and moment of inertia. Take the case of the frame shown in Fig. 1 as an example. We first construct Table I from which the denominators of Equations (13), (14) and (15) are determined and also the position of the elastic center and the slope m of the X-axis.

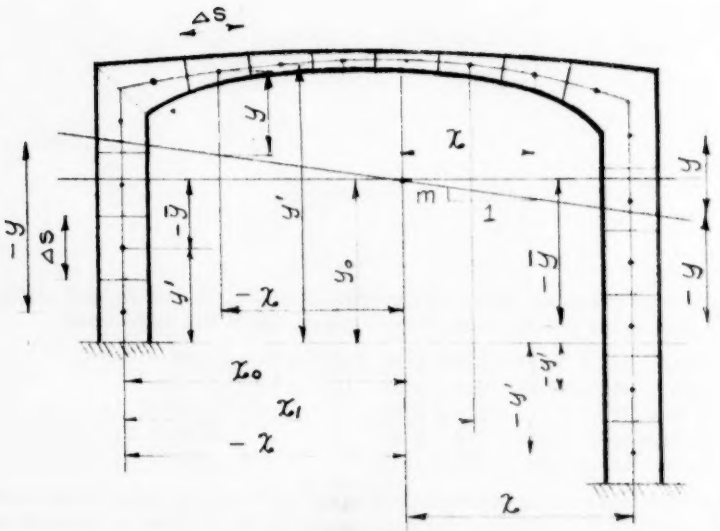

Δs
y
x
y
m
1
-y
Δs
-y
-x
y_o
x_o
x_1
-x
x

Fig. 1

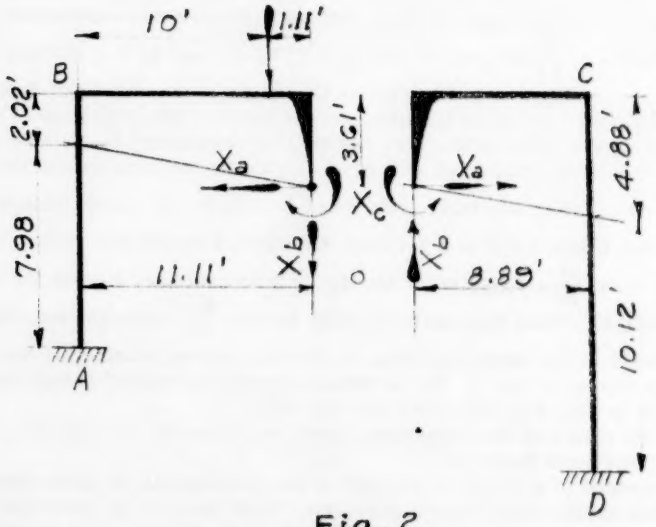
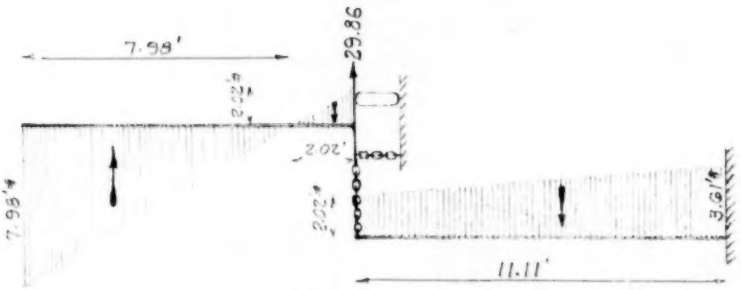

10'
1.11'
B
C
2.02'
3.61'
X_a
X_a
1
X_c
4.88'
7.98
X_b
X_b
11.11'
8.89'
10.12'
A
D

Fig. 2



Typical Conjugate Beam Diagram

Fig. 3

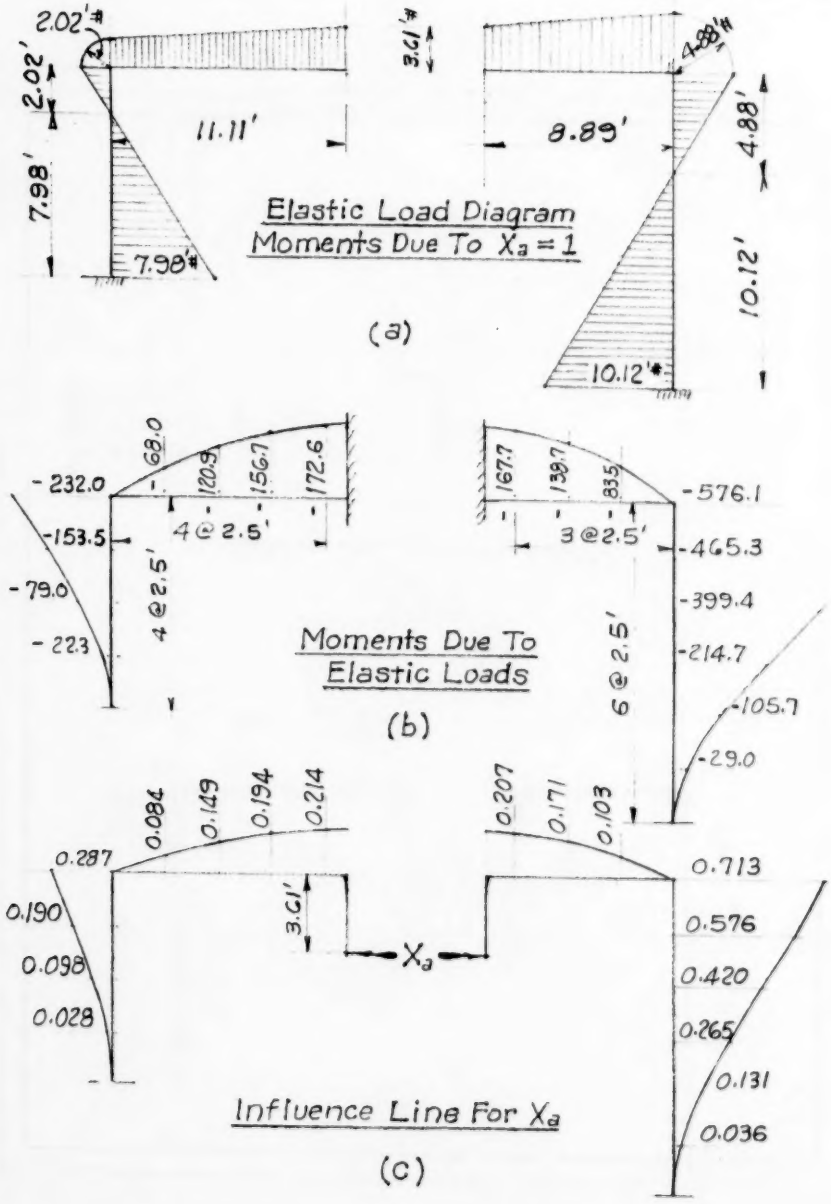
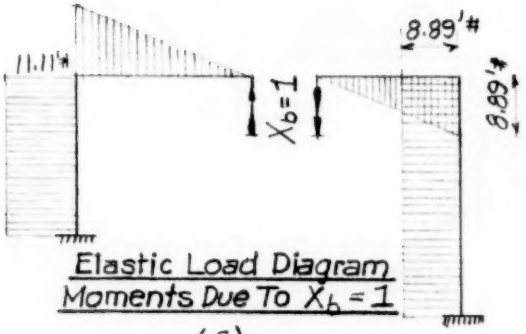
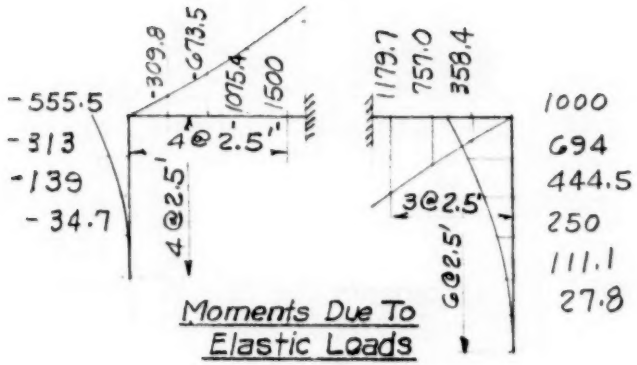

2.02'
2.02'
7.98'
11.11'
3.61'
8.89'
4.88'
4.88'
10.12'
7.98'
10.12'
Elastic Load Diagram
Moments Due To $X_a = 1$
(a)
-232.0
-153.5
-79.0
-22.3
68.0
120.9
156.7
172.6
4 @ 2.5'
4 @ 2.5'
Moments Due To
Elastic Loads
(b)
167.7
139.7
83.5
-576.1
-465.3
-399.4
-214.7
-105.7
-29.0
3 @ 2.5'
6 @ 2.5'
0.287
0.190
0.098
0.028
0.084
0.149
0.194
0.214
3.61'
X_a
0.207
0.171
0.103
0.713
0.576
0.420
0.265
0.131
0.036
Influence Line For X_a
(c)

Fig. 4



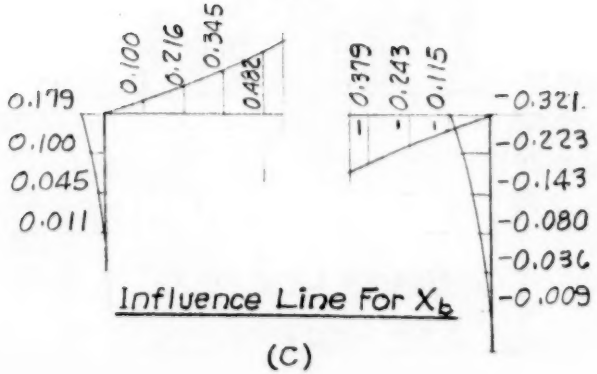
Elastic Load Diagram
Moments Due To $X_b = 1$

(a)



Moments Due To
Elastic Loads

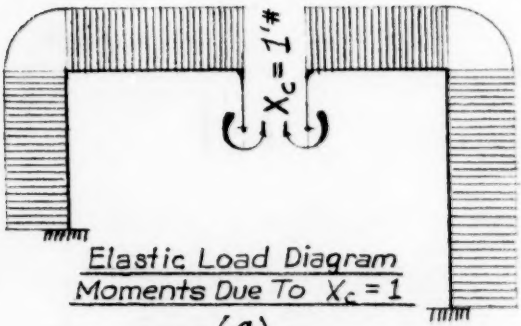
(b)



Influence Line For X_b

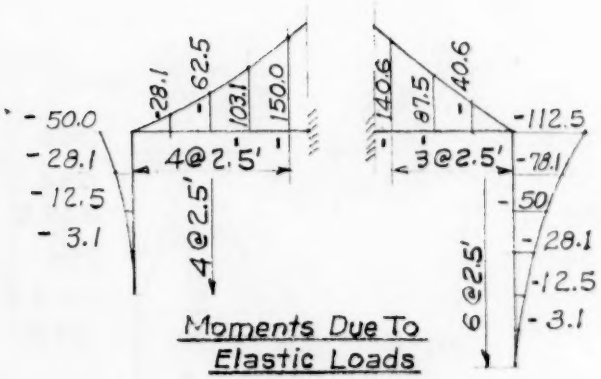
(c)

Fig.5



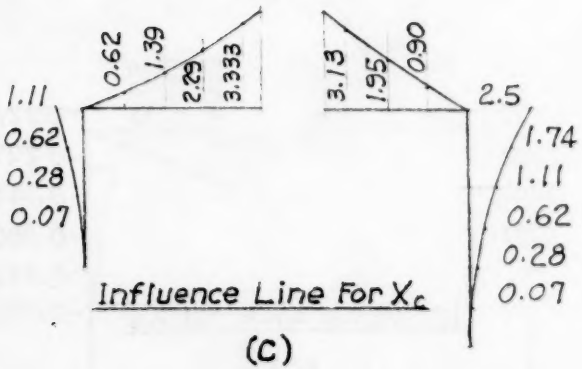
Elastic Load Diagram
Moments Due To $X_c = 1$

(a)



Moments Due To
Elastic Loads

(b)



Influence Line For X_c

(C)

Fig. 6

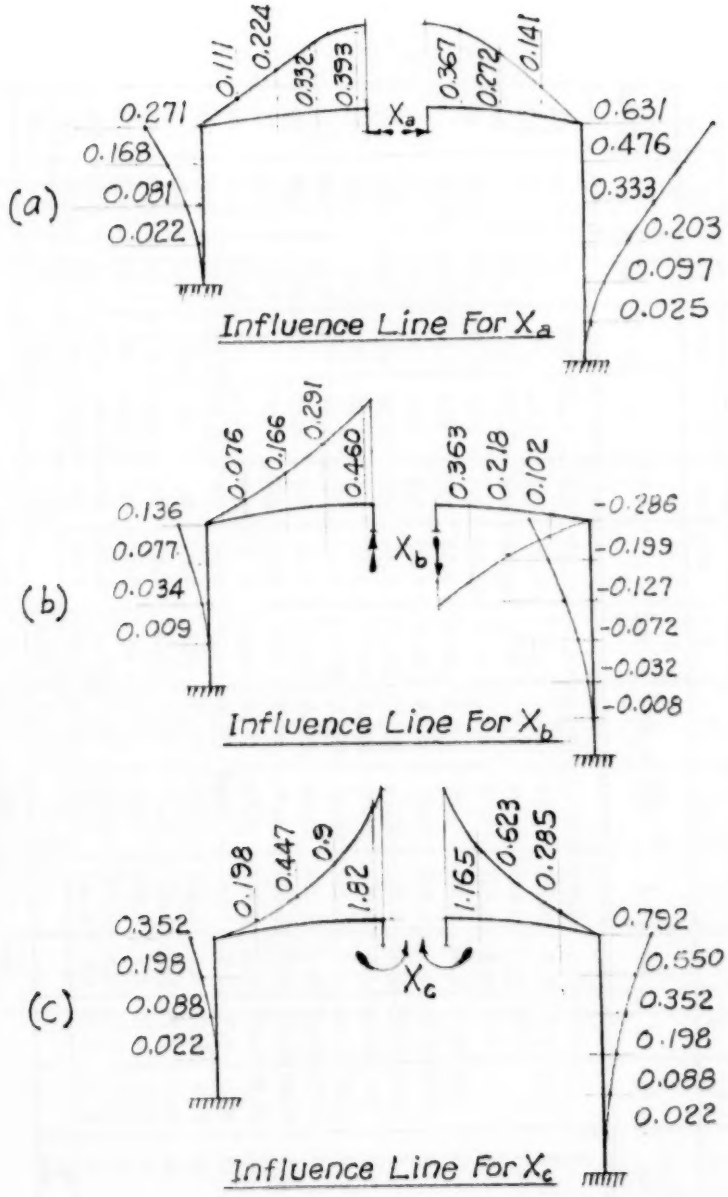

(a)
0.271
0.168
0.081
0.022
0.111
0.224
0.332
0.393
X_a
0.367
0.272
0.141
0.631
0.476
0.333
0.203
0.097
0.025
Influence Line For X_a
(b)
0.136
0.077
0.034
0.009
0.076
0.166
0.291
0.460
X_b
0.363
0.218
0.102
-0.286
-0.199
-0.127
-0.072
-0.032
-0.008
Influence Line For X_b
(c)
0.352
0.198
0.088
0.022
0.198
0.447
0.9
1.82
X_c
1.165
0.623
0.285
0.792
0.550
0.352
0.198
0.088
0.022
Influence Line For X_c

Fig. 7

TABLE I

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Sect. No.	x_1	y'	Δs	I	$\frac{\Delta s}{I}$	$x_1\frac{\Delta s}{I}$	$y'\frac{\Delta s}{I}$	x	$x^2\frac{\Delta s}{I}$	$\bar{y}$	$x\bar{y}\frac{\Delta s}{I}$	mx	y	$y^2\frac{\Delta s}{I}$
1		1.25	2.5	0.667	3.75		4.7	-10.35	402	- 8.44	327	-1.67	-10.11	384
2		3.75	2.5	0.667	3.75		14.1	-10.35	402	- 5.94	231	-1.67	- 7.61	217
3		6.25	2.5	0.667	3.75		23.4	-10.35	402	- 3.44	133	-1.67	- 5.11	98
4		8.75	2.5	0.667	3.75		32.8	-10.35	402	- 0.94	37	-1.67	- 2.61	25
5	1.25	10.25	2.55	0.667	3.83	4.8	39.3	- 9.10	317	0.56	- 20	-1.46	- 0.90	3
6	3.75	10.65	2.52	0.528	4.77	17.9	50.8	- 6.60	208	0.96	- 30	-1.06	- 0.10	0
7	6.25	10.95	2.51	0.083	30.24	189.0	331.1	- 4.10	508	1.26	-155	-0.66	0.60	11
8	8.75	11.15	2.5	0.051	49.01	428.8	546.5	- 1.60	125	1.46	-115	-0.26	1.20	71
9	11.25	11.15	2.5	0.051	49.01	551.4	546.5	0.90	40	1.46	64	0.14	1.60	125
10	13.75	10.95	2.51	0.083	30.24	415.8	331.1	3.40	349	1.26	129	0.55	1.81	99
11	16.25	10.65	2.52	0.528	4.77	77.5	50.8	5.90	166	0.96	27	0.95	1.91	17
12	18.75	10.25	2.55	0.667	3.83	71.8	39.3	8.40	270	0.56	18	1.35	1.91	14
13	20.0	8.75	2.5	0.667	3.75	75.0	32.8	9.65	349	- 0.94	- 34	1.55	0.61	1
14	20.0	6.25	2.5	0.667	3.75	75.0	23.4	9.65	349	- 3.44	-125	1.55	- 1.89	13
15	20.0	3.75	2.5	0.667	3.75	75.0	14.1	9.65	349	- 5.94	-215	1.55	- 4.39	72
16	20.0	1.25	2.5	0.667	3.75	75.0	4.7	9.65	349	- 8.44	-305	1.55	- 6.89	178
17	20.0	-1.25	2.5	0.667	3.75	75.0	- 4.7	9.65	349	-10.94	-396	1.55	- 9.39	330
18	20.0	-3.75	2.5	0.667	3.75	75.0	-14.1	9.65	349	-13.44	-487	1.55	-11.89	530
Σ					213.2	2207.0	2066.6		5685		-916			2188

$$x_o = \frac{2207}{213.2} = 10.35' \quad y_o = \frac{2066.6}{213.2} = 9.69' \quad m = -\frac{916}{5685} = -0.161$$

TABLE II

Left Cantilever

Sect. No.	$M_a = y$	$\frac{\Delta s}{I}$	$\frac{M_a \Delta s}{I}$	1	2	3	4	5	6	7	8	
1	-10.11	3.75	-37.9	-47.4	-142.1	-236.9	-331.6					
2	- 7.61	3.75	-28.55		- 35.7	-107.0	-178.3					
3	- 5.11	3.75	-19.16			- 23.9	- 71.8					
4	- 2.61	3.75	- 9.79				- 12.2					
			-95.4					-238.5	-477.0	-715.5	-954.0	
5	- 0.90	3.83	- 3.45					- 4.3	- 12.9	- 21.5	- 30.2	
6	- 0.10	4.77	- 0.48						- 0.6	- 1.9	- 3.0	
7	0.60	30.24	18.14							22.7	68.0	
8	1.20	49.01	58.82								73.5	
Σ				-47.4	-177.8	-367.3	-593.9	-242.8	-490.5	-716.2	-845.7	
x_a				0.022	0.081	0.168	0.271	0.111	0.224	0.332	0.393	

Right Cantilever

Sect. No.	$M_a = y$	$\frac{\Delta s}{I}$	$\frac{M_a \Delta s}{I}$	17	16	15	14	13	12	11	10	9
18	-11.89	3.75	- 44.6	-55.7	-167.3	-278.8	-390.3	-501.9	-613.3			
17	- 9.39	3.75	- 35.2		- 44.9	-132.0	-220.0	-308.0	-396.0			
16	- 6.39	3.75	- 25.8			- 32.3	- 96.8	-161.2	-225.8			
15	- 4.39	3.75	- 16.5				- 20.6	- 61.9	-103.1			
14	- 1.89	3.75	- 7.1					- 8.9	- 26.6			
13	0.61	3.75	2.3						2.9			
			-126.9							-317.2	-634.4	-951.6
12	1.91	3.83	7.3							9.1	27.4	45.7
11	1.91	4.77	9.1								11.4	34.2
10	1.81	30.24	54.7									68.4
9	1.60	49.01	78.4									
Σ				-55.7	-211.3	-443.1	-727.7	-1041.9	-1361.9	-308.1	-595.6	-803.3
x_a				0.025	0.097	0.203	0.333	0.476	0.631	0.141	0.272	0.367

TABLE III

Left Cantilever

Sect. No.	$M_b = x$	$\frac{\Delta s}{I}$	$\frac{M_b \Delta s}{I}$	1	2	3	4	5	6	7	8	
1	-10.35	3.75	- 38.8	-48.5	-145.5	-242.5	-339.5					
2	-10.35	3.75	- 38.8		- 48.5	-145.5	-242.5					
3	-10.35	3.75	- 38.8			- 48.5	-145.5					
4	-10.35	3.75	- 38.8				- 48.5					
			-155.2					-388.0	-776.0	-1164.0	-1552.0	
5	- 9.1	3.83	- 34.8					- 43.5	-130.5	- 217.5	- 304.5	
6	- 6.6	4.77	- 31.5						- 39.4	- 118.2	- 196.8	
7	- 4.1	30.24	-124.0							- 155.0	- 465.0	
8	- 1.6	49.01	- 78.4								- 98.0	
$\sum$				-48.5	-194.0	-436.5	-776.0	-431.5	-945.9	-1654.7	-2616.3	
x_b				0.009	0.034	0.077	0.136	0.076	0.166	0.291	0.460	

Right Cantilever

Sect. No.	$M_b = x$	$\frac{\Delta s}{I}$	$\frac{M_b \Delta s}{I}$	17	16	15	14	13	12	11	10	9
18	9.65	3.75	36.2	45.3	135.8	226.3	316.8	407.3	497.8			
17	9.65	3.75	36.2		45.3	135.8	226.3	316.8	407.3			
16	9.65	3.75	36.2			45.3	135.8	226.3	316.8			
15	9.65	3.75	36.2				45.3	135.8	226.3			
14	9.65	3.75	36.2					45.3	135.8			
13	9.65	3.75	36.2						45.3			
			217.2							543.0	1086.0	1629.0
12	8.4	3.83	32.2							40.2	120.7	201.2
11	5.9	4.77	28.1								35.1	105.4
10	3.4	30.24	102.8									128.5
9	0.9	49.01	44.1									
$\sum$				45.3	181.1	407.4	724.2	1131.5	1629.3	583.2	1241.8	2064.1
x_b				-0.008	-0.032	-0.072	-0.127	-0.199	-0.286	-0.102	-0.218	-0.363

TABLE IV

Left Cantilever

Sect. No.	$M_c = 1$	$\frac{\Delta s}{I}$	$\frac{M_c \Delta s}{I}$	1	2	3	4	5	6	7	8	
1	-1.0	3.75	- 3.75	-4.7	-14.1	-23.4	-32.8					
2	-1.0	3.75	- 3.75		- 4.7	-14.1	-23.4					
3	-1.0	3.75	- 3.75			4.7	-14.1					
4	-1.0	3.75	- 3.75				- 4.7					
			-15.0					-37.5	-75.0	-112.5	-150.0	
5	-1.0	3.83	- 3.83					- 4.8	-14.4	- 23.9	- 33.5	
6	-1.0	4.77	- 4.77						- 6.0	- 17.9	- 29.8	
7	-1.0	30.24	-30.24							- 37.7	-113.4	
8	-1.0	49.01	-49.01								- 61.3	
Σ				-4.7	-18.8	-42.2	-75.0	-42.3	-95.4	-192.0	-388.0	
x_c				0.022	0.088	0.198	0.352	0.198	0.447	0.9	1.82	

Right Cantilever

Sect. No.	$M_c = 1$	$\frac{\Delta s}{I}$	$\frac{M_c \Delta s}{I}$	17	16	15	14	13	12	11	10	9
18	-1.0	3.75	- 3.75	-4.7	-14.1	-23.4	-32.8	-42.2	- 51.6			
17	-1.0	3.75	- 3.75		- 4.7	-14.1	-23.4	-32.8	- 42.2			
16	-1.0	3.75	- 3.75			- 4.7	-14.1	-23.4	- 32.8			
15	-1.0	3.75	- 3.75				- 4.7	-14.1	- 23.4			
14	-1.0	3.75	- 3.75					- 4.7	- 14.1			
13	-1.0	3.75	- 3.75						- 4.7			
			-22.5							-56.2	-112.5	-168.8
12	-1.0	3.83	- 3.83							- 4.8	- 14.4	- 23.9
11	-1.0	4.77	- 4.77								- 6.0	- 17.9
10	-1.0	30.24	-30.24									- 37.8
9	-1.0	49.01	-49.01									
Σ				-4.7	-18.8	-42.2	-75.0	-117.2	-168.8	-61.0	-132.9	-248.4
x_c				0.022	0.088	0.198	0.352	0.550	0.792	0.286	0.623	1.165

We now compute the deflection, at the desired load points, of the statically determinate cantilevers, under the action of $X_a = 1$, $X_b = 1$ and $X_c = 1$. This is easily and rapidly done if the work is arranged in tabular form. Tables II, III and IV are an example of one arrangement of such tables. We divide the deflection, of the cantilevers, at the various load points by the denominator of the appropriate Equation (13), (14) or (15) to obtain the influence ordinates for X_a, X_b and X_c. These ordinates are shown in Tables II, III and IV and in Fig. 7.

Using these influence values, and placing a 1000 pound load at the center of the frame, one finds for the moments:

at A, 1820 + 393 x 11.36 + 460 x 10.35 - 1000 x 10 = 1044'#
B, 1820 + 393 x 1.36 + 460 x 10.35 - 1000 x 10 = 2886'#
℄, 1820 - 393 x 1.4 + 460 x 0.35 = 1431'#
C, 1820 - 393 x 1.86 - 460 x 9.65 = 3350'#
D, 1820 + 393 x 13.14 - 460 x 9.65 = 2547'#

Although the overall dimensions of the frame shown in Fig. 1 are the same as the frame treated by the author there are great changes in the values of the bending moments computed for the two frames. This difference is caused by the greatly varying moment of inertia of the frame treated by the writer and the fact that this frame has an arch-rise of 1.15 feet in the 20 feet span of the frame.

This general method of analysis is very practicable for use in analysing the stress in concrete or plate rib arches. In such cases one must add into Equations (4), (5) and (6) terms to represent the work done by the structure in resisting the direct stresses. The method can be quickly applied to the computation of the stresses in large hanger frames, highway or railway bridge arches(1) and continuous framed structures.

REFERENCE

1. See, for example, Continuous Tied Arch Built in Missouri by Howard H. Mullins, Engineering News-Record June 5, 1941 (Vol. p 896); Discussion of Influence Lines for Reactions of Continuous Trusses by Horward H. Mullins, Proc. Am. Soc. Civil Engrs., Nov. 1956 p. 1112-9 to 1112-35; and Discussion of Analysis of Multiple-Span Continuous Trusses by Howard H. Mullins, Proc. Am. Soc. Civil Engrs., Sept. 1957 p. 1382-45 to 1382-60.

R. E. BOWLES.[1]—In his paper, Mr. Zimmerman gives a precise solution of a single storey portal in the Hardy Cross method. This is ideal as an exposition of the classic method, but in his third paragraph he makes particular reference to the needs of practical or consulting engineers.

It is submitted that whilst the original method is fundamental, the development of moment patterns as devised[2] by Dr. Bolton[3] is preferable for ease of solution. A study of Bolton's method is recommended.

1. Lecturer, Structural Eng. Dept., College of Science & Technology, Manchester, England.
2. Paper read Jan. 24, 1952 before the Institution of Structural Engineers, London, England.
3. Lecturer, Dept. of Civ. Eng., Liverpool Univ., England.

As a demonstration, the original portal is solved below.

Line (a) is a pattern of the moments produced by a rotation of B of unknown amount.

Line (b) is a pattern of the moments produced by a rotation of C of unknown amount.

Line (c) is a pattern of the moments produced by an unknown horizontal displacement of B and C.

Line (d) represents 9/35 of (c) giving a shear equal in magnitude to that of (a).

Line (e) represents the sum of (a) and (d) and is a pattern for moments due to a rotation of B and a displacement giving no shear. This is referred to as a "no shear pattern."

Line (f) represents 12/35 of (c) giving a shear equal in magnitude to that of (b).

Line (g) represents the sum of (b) and (f) and is a pattern for the moments due to a rotation of C and a displacement giving no shear.

Normally (e) and (g) would be used to relax any unbalance at B and C e.g. ± 2,500, but in this example this balance involves equal changes at B and C. This can be effected by making:-

Line (h) equal to line (g) x -.291

Line (k) is the sum of (e) and (h) thus giving a no shear pattern for rotation of B and C together with displacement and equal but opposite moments at B and C. This is used to relax as shown in line (m).

	M_{AB}	M_{BA}	M_B	M_{BC}	M_{CB}	M_C	M_{CD}	M_{DC}	Shear
(a)	+2.00	+4.00	+6.00	+2.00	+1.00	+1.00	0	0	+6/10
(b)	0	0	+3.00	+3.00	+6.00	+14.00	+8.00	+4.00	+12/15
(c)	-9.00	-9.00	-9.00	0	0	-4.00	-4.00	-4.00	$-\frac{18}{10}-\frac{8}{15}=-\frac{7}{3}$
(d)	-2.314	-2.314	-2.314	0	0	-1.029	-1.029	-1.029	-6/10
(e)	- .314	+1.686	+3.686	+2.00	+1.00	-0.029	-1.029	-1.029	0
(f)	-3.086	-3.086	-3.086	0	0	-1.371	-1.371	-1.371	-12/15
(g)	-3.086	-3.086	- .086	+3.00	+6.00	+12.619	+6.629	+2.629	0
(h)	+ .899	+8.99	+ .025	- .874	-1.750	-3.683	-1.933	- .766	
(k)	+ .585	+2.585	+3.711	+1.126	- .750	-3.711	-2.961	-1.795	
(m)	+ 394	+1.742	+2.500	+ 758	- 505	-2.500	-1.995	-1.210	
Final Moments	+ 394	+1.742	0	-1.742	+1.995	0	-1.995	-1.210	

ROBERT B. HARRIS,[1] A.M. ASCE.—Mr. Zimmermann's paper should be welcomed by the practicing engineer looking for a rapid and reasonably accurate solution to some of his problems. It does seem that in the interests of economy of time and effort certain aspects of the solutions presented by the author have been overlooked.

In the Moment Distribution example given by Mr. Zimmermann he adheres to the bending moment convention throughout the solution. If a convention using clockwise moments acting on the end of a member is assigned a positive sign, the operations during the distributing process can be made with more rapidity. Also if the distribution is carried out joint by joint so that carry over factors are used immediately, the carried over moment becomes a part of the fixed end moment at adjacent joints. This allows the distribution series to converge more rapidly and saves considerable arithmetic.[2]

Another point should be made. The distribution factors should be expressed to the third significant figure to insure an accuracy commensurate with the accuracy of the results to be obtained and the usual design assumptions. Bents such as represented by the author are sensitive to wide variations due to these factors.

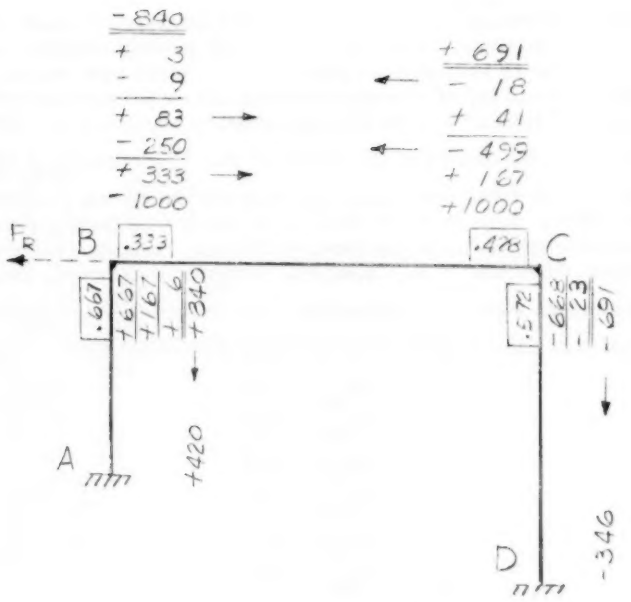
In most design problems moments are required for a variety of loading conditions such as dead load, operational loads, snow load, wind load and so forth. It is not necessary to solve distributions for each of these once a basic solution is obtained. If a fixed end moment of ±1000 foot lbs. is used the resulting moments are virtually percentage factors of any other symmetrical fixed end moments which may be derived from any condition of loading.

Figure H1 gives a soltuion with a ±1000 ft. lbs. fixed end moment with the bent restrained from sidesway. The carry over for the columns need not be completed until all moments have been balanced between B and C since no loading is acting on these columns. The restraining force has been labeled F_R. Balancing began at point B and one half of the balancing moment was carried to joint C to be included with the fixed end moment at C. A line has been used after a balancing operation to signify that this has been accomplished. Note that the sign has been retained in the carry over. The free bodies of each of the members are shown below the distribution and the restraining force, F_R, is calculated by simple statics. This value will be noted as being 56.9 lbs. for the fixed end moment of ±1000 ft. lbs. The restraining force for the author's given case of ±2500 ft. lbs. is therefore

$$F_R = \frac{56.9}{1000} \times 2500 = 142 \text{ lbs.}$$

The author's notation in Figure 3 of approximate distances to points of inflection are approximate only to the extent that the base of the bent may be considered fixed. An assumption of fixity at the end of a prismatic member results in an induced moment at the fixed end of one half the moment applied at the other unless altered by intervening loads and the inflection point is, by proportion, at one third the length of the member. Therefore in the example

1. Assoc. Prof. of Civ. Eng., Univ. of Michigan, Ann Arbor, Mich.
2. "Statically Indeterminate Structures" by L. C. Maugh, 1946, John Wiley and Sons, New York, p. 62.



$$F_R - H_1 - H_2 = 0$$

$$F_R = 126.0 - 69.1 = 56.9$$

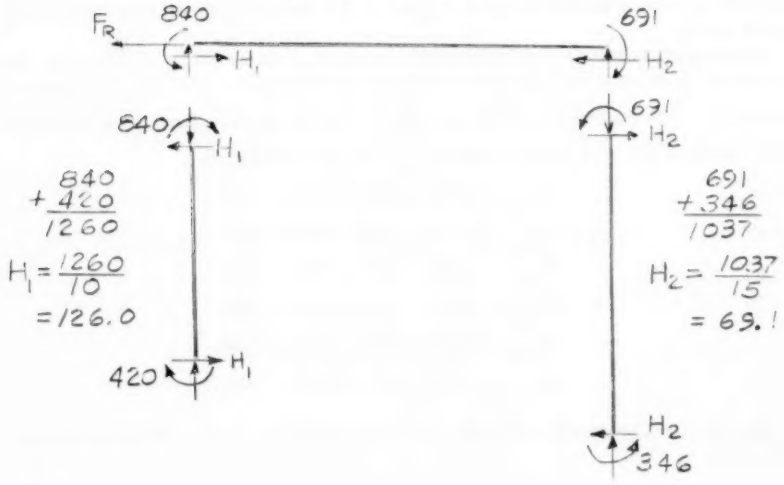


Figure H1

the distances are exactly 3.33 feet and 5.00 feet respectively without regard to the values of the moments.

Moment distribution can only adjust for the rotations of elastic tangents at the ends of the members, evaluated in terms of moment couples, hence it is frequently easier to assume an angular rotation of the joint caused by sidesway rather than to try to evaluate rotations caused by assumed deflections.

An assumed rotation of joint B represented by a moment of -1000 ft. lbs. is applied to the top of column BA in Figure 16. The stiffness $\frac{I}{L}$ of BA is 6.66 and of CD is 4.44. The compatible moment for the top of column CD is therefore -444 ft. lbs. Since the bases at A and D have been assumed fixed, these moments also apply at the lower ends of the columns. Figure H2 gives the solution. The value of F_B causing the assumed rotations is 148.9 lbs. acting toward the right. The correcting ratio is then $\frac{142}{148.9} = 0.955$ and the end moments due to sidesway in the example are as follows:

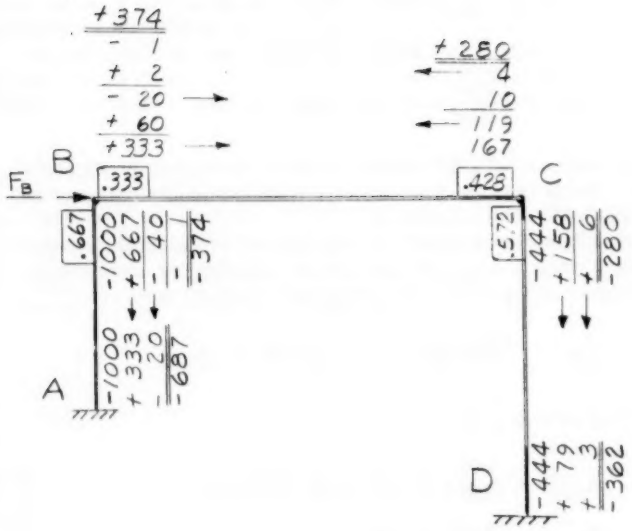
$$M_{AB} = -655$$
$$M_{BA} = -357$$
$$M_{BC} = +357$$
$$M_{CB} = +267$$
$$M_{CD} = -267$$
$$M_{DC} = -346$$

These are not the same as given by the author in Figure 5 because he has a sign error in Figure 4 and because he has not carried his process to the degree of convergence of Figure H2. Making the sign correction and balancing the system a fourth time in Figure 4, the author's moments agree with those above.

If the end moments are computed for the sidesway force of 56.9 lbs. the resulting moments can be added to those of Figure H1 and the final moments obtained. The correcting ratio is $\frac{56.9}{148.9} = 0.382$ and the final end moments are as follows for an initial 1000 ft. lb. fixed end moment.

$$M_{AB} = +420 - .382 \times 687 = +157$$
$$M_{BA} = +840 - .382 \times 374 = +697$$
$$M_{BC} = -840 + .382 \times 374 = -697$$
$$M_{CB} = +691 + .382 \times 280 = +798$$
$$M_{CD} = -691 - .382 \times 280 = -798$$
$$M_{DC} = -346 - .382 \times 362 = -484$$

The final end moments for the given (+)250 ft. lb. fixed end moment are therefore



687
+374
10 | 1061
106.1 to right

362
+280
15 | 642
42.8 to right

F_B = 106.1 + 42.8 = 148.9 lbs. to right

Figure H2

$$M_{AB} = + \frac{157}{1000} \times 2500 = +392 \text{ ft. lbs.}$$

$$M_{BA} = + \frac{697}{1000} \times 2500 = +1,740 \text{ ft. lbs.}$$

$$M_{BC} = - \frac{697}{1000} \times 2500 = -1,740 \text{ ft. lbs.}$$

$$M_{CB} = + \frac{798}{1000} \times 2500 = +2,000 \text{ ft. lbs.}$$

$$M_{CD} = - \frac{798}{1000} \times 2500 = -2,000 \text{ ft. lbs.}$$

$$M_{DC} = - \frac{484}{1000} \times 2500 = -1,210 \text{ ft. lbs.}$$

These are in agreement with the values given by the author in Figure 6, after corrections. From these end moments the shears in the columns and the horizontal member can be computed.

To further illustrate the usefulness of this procedure consider a second condition. Assume a snow load of 600 lbs. per foot uniformly distributed across the top member of the bent. The fixed end moments are

$$\pm \frac{wL^2}{12} = \pm \frac{600 \times 20 \times 20}{12} = \pm 20,000 \text{ ft. lbs.}$$

and the final end moments are therefore

$$M_{AB} = + \frac{157}{1000} \times 20,000 = +3,140 \text{ ft. lbs.}$$

$$M_{BA} = + 13,940 \text{ ft. lbs.}$$

$$M_{BC} = - 13,940 \text{ ft. lbs.}$$

$$M_{CB} = + 15,960 \text{ ft. lbs.}$$

$$M_{CD} = - 15,960 \text{ ft. lbs.}$$

$$M_{DC} = - 9,680 \text{ ft. lbs.}$$

Thus having solved the frame for its basic distributions all similar distributions made be obtained very rapidly and with minimum error.

ALBERT D. M. LEWIS,[1] A.M. ASCE.—In this paper the author has made a very interesting parallel presentation of three well-known methods for solving a rigid frame. However, the statement near the end of the paper to the effect that errors in arithmetic and algebraic signs must be avoided in the slope deflection method is also applicable to the other two methods. In fact in the moment distribution solution which was presented, it is extremely important that arithmetic and signs are correct because the mistakes may not be detected. If a mistake is made in performing a carryover to joint B, for example, the two final moments obtained at joint B will still be equal even though incorrect. The fact that the sum of the vertical shears equals the vertical load is a check only in regard to the computation of the shears. Any

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pair of moments can be substituted in Equations 10 and 10A and the sum of the shears so obtained will equal the vertical load.

In the slope deflection method, if the sum of the final moments at each of the joints is zero, a check is provided on the formation of the joint equations, the solution of the resulting simultaneous equations, and the computation of the final moments. It is very unlikely that mistakes made after the writing of the moment expressions will go undetected. The labor of solving the simultaneous equations can be reduced by applying the approximation procedures developed by Goldberg.(1)

The mistakes in carryover in moment distribution can be detected by computing the rotations of the members at each joint in the manner described by Wang and Eckel(2) and noting whether or not they are equal.

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DEAS ISLAND TUNNEL[a]

Discussion by James I. Rooney and A. A. Eremin

JAMES I. ROONEY,[1] M. ASCE.—This excellent paper on the Deas Island Tunnel is timely due to the great number of subaqueous vehicular tunnels which have been built in the current era and are contemplated.

Up to recent times on the American continent invariably the cross section of subaqueous tunnels simulated a circular pattern, even those lately constructed, such as the Lincoln Tunnel, Baltimore Harbor Tunnel, Hampton Roads Tunnel, and the Baytown and Pasadena Texas Tunnels. However, the rectangular box-shaped section has now come into the picture, with the advent of the Belle Chase and Harvey Canal Tunnels in Louisiana, Lachine Canal Montreal Tunnels, Havana Tunnels, and the Deas Island Tunnel.

Although the construction methods used for sinking and placing the prefabricated sections for the Deas Island and the Havana Harbor Tunnels have followed the Rotterdam Tunnel pattern closely, it might be of interest to note that many of the construction features employed were initially developed in placing the Posey Tube. Further, as in the case of the Deas Island Tunnel, no steel protective covering was used on the Posey Tube.

Apparently complete comparative cost studies were made of alternate bridge and tunnel types of construction for the Deas Island site. These studies indicated that the proposed tunnel, as now being constructed, would run some 15 percent lower in cost than a bridge alternate. These findings are of particular interest to the writer since for similar major river crossings in this country, even where severe navigation clearance restrictions have applied and poor ground conditions existed for bridge foundations, the estimated costs for tunnel alternates invariably have run considerably higher than for bridge types.

Limited information has been presented concerning the mechanical ventilation system for the tunnel. It would appear that with only one duct being provided alongside each roadway, a longitudinal system is to be used, with fresh air being blown in over the traffic entering half of the tunnel length, and exhausted over the outgoing half of the tunnel length. Further information in the paper would help to show how this important safety feature functions.

On the recently completed Harvey Canal Tunnel, about 1,100 feet long between portals, the design was developed so that it could be operated on a completely automatic basis with no personnel required at the tunnel site. This paper states that the Deas Island Tunnel will be equipped for automatic operation. It would be of interest to know whether features, such as ventilation,

a. Proc. Paper 1436, November, 1957, by Per Hall, Troels Brøndum-Nielsen and H. R. Kivisild.

1. Highway Tunnel Entr., U. S. Bureau of Public Roads, Washington, D.C.

lighting, and traffic signals, will be automatically controlled, and if any personnel will be required at the site.

The authors are to be commended for their very interesting and comprehensive paper. Due to the rather critical foundation conditions at the tunnel site, the detailed description of the investigations made and the construction methods used should be of interest to tunnel engineers.

A. A. EREMIN,[1] A.M. ASCE.—By showing methods of design and construction of the Deas Island Tunnel, the authors made a valuable contribution to engineering. Recently the open-trench type of subaqueous tunnel has found wide application, yet more study and information on details are required for further development of this method of tunneling. The reinforced concrete subaqueous sections of the Deas Tunnel may be compared with the reinforced concrete sections of the Posey Tube under Eastuary between Oakland and Alameda, California, which was constructed in 1928 and described by the writer in his recent publication.(2)

During design of the Posey Tube, an alternate design of double steel shell type was detailed and enclosed in the bids. However, the bids have shown that the reinforced concrete design was more economical, as was revealed in the design ot the Deas Island Tunnel.

The stiffness factor of the subaqueous section of the Deas Tunnel or the ratio of the depth of tunnel section to the tunnel length between the portal buildings is approximately the same as that in the Posey Tube. Therefore, obviously the rigidity of the Deas Tunnel comparing with the Posey Tube may be improved only with the increased rigidity of the foundation and the tunnel cover.

A notable feature of the tunnel cover used in the Deas Tunnel is application of the reinforced concrete articulated mattresses over the tunnel cover. It is well known that the tunnel cover may not only affect the reducing of the tunnel tidal grade fluctuation, but also serve as protection of the tunnel structure from accidents, which possibly could be caused by fallen anchors during shipping over the stream. In the reinforced concrete mattresses used in the cover of Deas Tunnel, it is interesting to note the detail of seal at the articulations.

By using the tunnel box type sections in the Deas Tunnel, the tunnel length between the portal buildings was reduced. Likewise, the length of the Posey Tube at the Oakland approach was reduced by changing the circular tube section to a rectangular box type tunnel section. However, in regards to distribution of stresses and strains, the circular section of tunnel has greater advantages of uniform variation of stresses.

In conclusion, the writer wishes to express that it would be interesting to have more details and information on the settlement of tunnel sections during placing operations and unwatering of the tube.

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2. "Tunnels, Underground Structures, and Air Raid Shelters" by A. A. Eremin, 1958. "Civil Engineer," March, 1958, p. 114.

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1. Assoc. Bridge Engr., California State Bridge Dept., Sacramento, Calif.

ANALYSIS OF HELICAL BEAMS UNDER SYMMETRICAL LOADING[a]

Discussion by A. S. Hall and Victor R. Bergman
Corrections by Alan M. C. Holmes[1]

A. S. HALL,[2] A.M. ASCE.—Towards the end of this very useful article Mr. Holmes refers to the symbol J as a Modified Polar Moment of Inertia. The term "Torsion Constant" is usually used to describe J, and has the merit of avoiding any suggestion that J is related to I_p. Although the author of this paper clearly does not imply any real affinity between the Torsion Constant and the Polar Moment of Inertia for a given section, such a confusion is fairly widespread (due to their having the same value in the case of a circular section).

The writer feels it is in the interest of clarity to keep the identity of these two quantities clearly separate. For this reason the writer would express a plea for the use of the term "Torsion Constant" when referring to J.

CORRECTIONS.—The author wishes to make the following corrections:

Page 1437-1, second paragraph, "virogous" should read "rigorous."

Page 1437-4, twice on this page and wherever it appears thereafter, (pp. 1437-22, 1437-35)

δ should read γ

Page 1437-14, immediately after equation 46 the text should read:

"In the case of a helix with h - zero = θ, (a RING BEAM),"

A qualifying remark concerning equations 69, 70 and 71 is necessary:

These equations as they stand apply only to the upper half of a helical beam with a concentrated load at the center.

Values of M_n' for negative values of ϕ will be incorrect because $(\sin\phi)$ is not symmetrical about the midpoint. Values of M_n' for negative values of ϕ must be obtained by inserting positive values of ϕ in the equations.

Values of M_t' and M_b' for negative values of ϕ will be incorrect because $(1-\cos\phi)$ is not anti-symmetrical about the midpoint. Values of M_t' and M_b' for negative values of ϕ must be obtained by inserting positive values of ϕ in the equations and multiplying the total result by minus one.

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a. Proc. Paper 1437, November, 1957, by Alan M. C. Holmes.

1. Structural Designer, Skidmore, Owings & Merrill, Chicago, Ill.
2. Assoc. Prof. of Civ. Eng., New South Wales Univ. of Technology, Sydney, Australia.

In subsequent solutions, all integrals were integrated over the half arc (upper half), so that they are unaffected by the above remarks.

Equations 77 and 78 should be corrected to read as follows:

$$\Delta_{zo} = \int_o^{\alpha} \frac{M_n}{EI} \cdot \frac{\partial M'_n}{\partial \frac{W}{2}} ds + \int_o^{\alpha} \frac{M_t}{GJ} \cdot \frac{\partial M'_t}{\partial \frac{W}{2}} ds + \int_o^{\alpha} \frac{M_b}{EK} \cdot \frac{\partial M'_b}{\partial \frac{W}{2}} ds$$

Equation 77

This can eventually be reduced to the following form:

$$\Delta_{zo} = \frac{Kd}{\cos^2\theta} \frac{wr^4}{EI} = 2\, Kd' \frac{wr^4}{EI}$$

Equation 78

If equation 78 is stated in this form, the values of K_d' given in TABLE 2 are correct as they stand.

Page 1437-22 The last term of equation 79 should read:

$$+9.58C_2C_4 \text{ instead of } +9.58C_2O_4$$

VICTOR R. BERGMAN,[1] A.M. ASCE.—This paper by Mr. Holmes implicitly attests to the growing popularity in the United States of the helicoidal staircase. Even a brief study of the paper reveals internal evidence that a great amount of time and thought must have been devoted to its preparation and presentation. The author is to be thanked especially for his clarifying exposition of the role played by binormal moment and shear in reducing the normal bending moment and the torsional moment below the values computed by the simpler, but less accurate, approximate method of Ref. (10).

However, the writer feels impelled to question the "exactness" of even such an extended analysis as Mr. Holmes presents. It must be remembered that the theories for both flexure and torsion (reflected in the author's EI/GJ) were developed specifically for straight members. The helicoidal staircase cannot be termed even approximately straight: it is, instead, a very sharply curved member. Staircases of this type have been built with centerline radii no greater in length than the width of slab, and a radius equal to only one and one-half times the slab width is common.

That the sharp curvature of the slab does have an effect on the analysis is readily recognized when we consider, for example, the author's Eq. 52,

$$\psi_{no} = \int_o^{\alpha} \sin\theta \cdot \sin\phi \cdot \frac{M_b}{EA} ds ,$$

in which M_b = moment about a binormal axis, and
K = moment of inertia about the binormal axis.

1. Vice-Pres., Godwin Constr. Co., New York, N. Y.

In this equation, M_b/EK is intended to represent the change of angle (due to M_b) between two radii a unit distance apart measured along the longitudinal axis of the slab. Although this expression is employed quite justifiably in arch analysis where the depth of the arch ring is small compared with the radius of curvature, its use in connection with an "exact" analysis of a sharply curved member is questionable. Textbooks on strength of materials treat the bending of such sharply curved members as a special problem in which it is shown that straight line variation of stress is impossible because of the variations in fiber lengths.

The deviation from straight line stress distribution causes an increase of some 52% in the maximum stress value over that computed by the ordinary Mc/I formula, when the width of the slab equals its radius.(A) The effect of the slab's curvature on the angular change for a given bending moment is not as great as the effect on maximum stress: nevertheless, the use of an unmodified M_b/EK for the computation of change in angle cannot be expected to yield "precise" results.

In view of the fact that the basic equations employed by Mr. Holmes cannot be described as "exact" when applied to other than straight or gently curved members, can his solution properly be termed "a precise mathematical analysis"?

The sharp curvature of the usual helicoidal slab is responsible for still another source of error—that involved in assuming that uniform loading acts as a line loading along the longitudinal center line of slab. Actually, the center line of loading lies outside of the center line of slab, i.e., the loading has a certain eccentricity with respect to the center line of slab. This is readily seen if, on a plan view, we consider an element of slab area bounded by two adjacent radii. The element of area is trapezoidal and its c.g. (and that of the superimposed uniform loading) does not lie on the longitudinal center line of the slab. A simple calculation shows that the eccentricity, e, of the load is equal to $\frac{1}{12}\frac{b^2}{r}$, or, $e/r = \frac{1}{12}\left(\frac{b}{r}\right)^2$.

For a given slab width, the value of e/r rapidly decreases as the radius of curvature increases. However, for the sharp curvatures involved in the structures under consideration, the effect of the eccentricity of loading is not altogether negligible: the effect can, and perhaps should, be included in an "exact" analysis. One such, by Dr.-Inc. Wilhelm Fuchssteiner,(B) does take this effect into account. Incidentally, in a supplementary article published by Doctor Fuchssteiner in conjunction with H. S. Gedizli,(C) a table of values is furnished which greatly facilitates the solution of the equations for the shears and moments. (A mistake in Ref. C sets $a = I_x/I_y = b^2/d^2$: this should read, $a = I_x/I_y = d^2/b^2$.)

The foregoing remarks are not intended to disparage Mr. Holmes' admirably presented treatment of a complicated problem. What the writer does wish is to emphasize that certain "built-in" errors exist in the analytical treatment of a sharply curved helicoidal slab as a beam. Possibly, a truly precise solution can be achieved only by analyzing the stair slab as a curved, warped plate. On the other hand, the "exact" method advanced by Mr. Holmes may be sufficiently accurate to be entirely acceptable from the pragmatical viewpoint of the structural designer. However, the evaluation by purely mathematical methods of the errors involved would be very difficult.

Mr. Holmes can render further service to his readers by reporting and

analyzing the "model tests now in progress," which he mentions in his "Conclusions." It would indeed be both gratifying and reassuring to learn from model tests that the sources of error mentioned by the writer affect the validity and accuracy of the "exact" solution to only a relatively small degree.

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C. H. S. Gedizli & Wilhelm Fuchssteiner, "Zum Thema: 'Die freitragende Wendeltreppé," in "Beton Und Stahlbetonbau," Sept. 1955, p. 247.

TENTATIVE RECOMMENDATIONS FOR PRESTRESSED CONCRETE[a]

Discussion by A. W. Coutris and T. Y. Lin

A. W. COUTRIS,[1] J.M. ASCE.—A valuable aid for the advancement of linear prestressed concrete structures has appeared by the publication of this first recommended practice.

The collection and critical examination of a vast amount of scattered technical information, indispensable for an undertaking of this nature, is not a simple task by any means. The committee members are to be congratulated for their intensive and persevering efforts to express concisely the present state of our knowledge in this domain. Their work, differing from that of other committees concerned with structural design by being recommendatory and not mandatory, makes some brief remarks on the original purpose of such authoritative documents highly desirable.

The first authoritative body, organized to consider and pass upon newly acquired knowledge and information gleaned from experience, was the First Joint Committee on Concrete and Reinforced Concrete (1904-1917). Their Final Report of 1917(1) exerted a profound influence and served as a model for succeeding committees. Though various portions have been decidedly modified or amplified in the way of new assumptions, tabular values and formulas, its essential form has been retained.(2)

Most unfortunately, its advisory spirit was not retained. The Second Joint Committee decided that conditions would be improved by the dogmatic statements of a specification (1925), which established the pattern for codes in use today.

An intransigent attitude, however, does not suffice for a clear understanding of complicated structural behavior. Rational explanations of the complex behavior of concrete, as a constituent structural material in prestressed or reinforced construction, have drawn the widespread attention which they deserve only within the last decade. Similarly, the general necessity for an accurate transition from the strength of laboratory specimens to the strength of full scale structural elements has been clearly revealed during this period of time, and the need for a new specialized branch of scientific endeavor is forcefully beginning to manifest itself.(3) The same past ten years have witnessed a careful re-examination of the basic concept of the "safety factor,"(4) but its important consequences for design must be relegated to a remote future. Investigations of the inelastic character of continuous prestressed frames have barely occupied these last five years.(5,6)

In the face of the limitations of present knowledge no set of regulations can

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a. Proc. Paper 1519, January, 1958, Report of the Joint ACI-ASCE Committee on Prestressed Reinforced Concrete.

1. Cons. Engr., Paris.

hope to offer instructions which adequately cover every situation confronting designers. Moreover, the succinct embodiment of all existing analytical, experimental and practical evidence in a code is obviously unfeasible. These statements maintain their vigor even if the field is restricted to so-called "standard structures," i.e., structures for which specifications are primarily directed because they occur so frequently. Their validity is confirmed by the failures(7) which happen occasionally despite the fact that mandatory procedures are correctly employed.

Accordingly, the reaffirmation of the initial aim of authoritative documents, as fundamentally advisory reports to be used by competent engineers, seems particularly well justified for prestressed concrete. Valid deviations from their recommendations on the basis of specialized experience and sound judgment may be demanded under certain circumstances.

Fixed rules cannot produce or supersede intelligence, judgment and experience. On the contrary, intelligence, judgment and experience should control the interpretation and application of rules.

The writer is pleased that the First Committee on Prestressed Reinforced Concrete has recognized this fact by an earnest appeal to prevent the thoughtless execution of formulistic operations,(8) and believes that the interests of the public and engineering profession will continue to be served best only by reports preserving the advisory spirit which has been initiated.

The following comments on two of the more delicate issues are made in this same tone.

The first of these deals with friction loss in post-tensioned steel (208.2.1). The suggested values for wobble effect K and curvature effect μ relative to steel cables in metallic sheathing represent a conservative view point. On the basis of experimental results,(9) a K of 0.0005 for heavy gage tubing and 0.001 for light gage tubing is indicated. Two of these experiments with light gage tubing (which showed the much higher values of 0.003, 0.005) lack practical meaning, since common sense would guide a designer away from the use of this type of sheath in instances of the heavy vibration of the concrete. Recent field measurements(10,11) tend to confirm the use of 0.001 for K in an advance estimate of friction losses as reasonable.

Though the recommended μ values are more realistic, it is felt that they could be lowered somewhat with safety. For the past several years, the writer has seen employed and has himself employed a μ range of 0.15 - 0.35 and K range of 0.0004 - 0.0013, choice of specific values depending on the peculiarities of the project being treated. Every structure has performed most satisfactorily. In those cases where field tests(12) have been undertaken, a close agreement between predicted losses and measured losses nearly always occurred.

Regarding the coefficients, advised for high strength bars, considerable care should be exercised in their employment. Bars are very sensitive to any wobble in the duct due to their much greater stiffness compared with high tensile wire. Unfortunately, reported information is meager, but one source(13) does offer (0.001, 0.30), (0.0005, 0.30), (0.0000, 0.30) for steel ducts, exceeding the bars by 1/8, 1/4, 1/2 inches, respectively, and lubricated with a lead coating. Therefore, it may be suspected that the suggested design values for K of 0.0002, 0.0003 and for μ of 0.15, 0.20 incline toward excessive liberality. A K between 0.0005 and 0.001 and a μ between 0.20 and 0.30 seem more prudent. Their use is urged until this question is clarified by a large number of tests.

Actually, it should be emphasized that high friction losses are not normally a matter to worry about because efficient techniques exist for their reduction. These are:

1. Redesigning with simpler tendon profiles.
2. Rigid ducts for straights.
3. Flexible ducts for curves, enabling the formation of a natural catenary with a negligible wobble.
4. Jacking from both ends.
5. Building with a decent degree of field competence. Even in conventional construction, material cannot be placed and erected without some precautions.
6. Lubricated tendons and ducts.[14] Friction may be reduced to a minor aspect.[15]

Another interesting case, clearly revealing the powerful potential of lubricants, involves large circular tanks. In the past this type of structure has posed the problem of unusually high friction losses quite difficult to avoid. The encouraging results of recent experimental studies[16] with an oil possessing special properties corroborate strongly the effectiveness of this general approach.

In the event that none of these past six suggestions can be applied, a thorough economic analysis may discover occasionally that the expense of a relatively large increase in steel is small in contrast to the in-place cost of the tendons. Incidentally, this is generally true for moderate increases.

The second topic inviting discussion is that of ultimate flexural strength (209.2.1 a).

An important start has been made by the experimental research[17] which was taken as a base for the material is this section. The conception of a testing program, which (1) must cover a large enough range of relevant variables, (2) must test pieces capable of being handled by laboratory machines, (3) must yield information of immediate practical utility for design, strikes a natural difficulty. It may be that the successful accomplishment of all of these goals would necessitate a special test series for each one of them, a course of action made impossible due to limited resources. The restrictions which are placed upon impartial experimenters should be fully appreciated by any engineer before he assumes a critical position. The ensuing remarks are not meant as criticisms but are raised as unsettled points in need of further study.

First, it is reasonable to believe that expressions of ultimate flexural strength, derived for rectangular beams, are applicable to flanged sections if the neutral axis lies within the flange. But since they contain empirical coefficients, reliable predictions are obtained only with caution. The investigations[18] previously cited included a helpful chart, showing the effect of $k_1 k_3$ on the ultimate moment Mu. For Es P/f'_c below about 25, $k_1 k_3$ has a negligible influence on Mu. With a sharp increase in Es P/f'_c the sensitivity of the ultimate moment to variations in $k_1 k_3$ becomes strong. Therefore, experimental data, proving $k_1 k_3$ independent of the shape of the cross section and of b/d[19] for large Es P/f'_c would be reassuring. Observe the interesting fact that flanged beams, formed in practice by the insured interaction between slabs and their supporting beams, do not seem to be cause for alarm. Though their cross sections are most certainly not rectangular and upper bounds of b/d as high as 1.5 are sometimes attained, the wide b width of the interacting slab usually forces Es P/f'_c below 25, eliminating any dependence of the ultimate moment on $k_1 k_3$.

Second, the awkward introduction of three empirical parameters, k_1, k_2, k_3, is disconcerting. Most designers will not be in a position to undertake tests for these parameters. More conclusive data, demonstrating a reasonably consistent variance of $k_1 k_3$ with concrete strength f'_c, would inspire more confidence in the recommended value of 0.6 for $k_2/k_1 k_3$. The tests(20) mentioned above intimate 0.5. Are there other data in support of 0.6. One of the fundamental weaknesses of testing, an inability to proceed from the compressive strains of a relatively simple test to the flexural compressive stresses in a structural specimen, can hardly be revealed more clearly. Efforts to remove this weakness should be intensified, and direct attacks on the compressive stress block should be attempted.

Another line of thought,(21) valuable for the design office, was presented recently, introducing different dimensionless variables. An approximate formula for the ultimate moment of bonded members was developed, giving somewhat different results from that of section 209.2. The equations which were derived and coupled with a plastic analysis predicted the ultimate strength of experimental continuous beams with a good approximation. It was felt that moment redistribution should be considered in design, an opinion differing from that of section 213.4. Several factors aiding this redistribution were listed.

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7. See, for instance, E. Cohen. "Rigid Frame Failure Raises Questions as to Standard Design Assumptions." Civil Engineering. Vol. 26. February 1956. p. 90.
8. It should be noted that the philosophy behind the modern building code never intended to cause this effect. See F. Kerekes. "Review of changes in A.C.I. Building Code Requirements for Reinforced Concrete." Journal A.C.I. Vol. 27. September 1957. p. 188.
9. E. H. Cooley. "Friction in Post-tensioned Prestressing Systems." Research Report 1. October 1953, p. 17. Cement and Concrete Association, London.

10. E. G. Trimble. "Site Determination of Friction in Prestressing Cables and Interpretation of Test Data." Magazine of Concrete Research. Vol. 8 Number 23, August 1956.

11. E. G. Trimble. Discussion of "Site Determination of Friction in Prestressing Cables and Interpretation of Test Data." Magazine of Concrete Research. Vol. 9. Number 25, March 1957. p. 50.

12. To be published elsewhere.

13. E. H. Cooley. "Friction in Post-tensioned Prestressing Systems." Research Report 1. October 1953, p. 20. Cement and Concrete Association, London. (While a sufficiently large excess of duct over tendon obviously eliminates the wobble problem, objectionable consequences may be incurred, such as (1) higher duct costs, (2) increased height and width and (3) added dead weight, due to enlarged overall section dimensions.

14. M. J. Montagnon. "Aspects pratiques de la Precontrainte par cables. Le Problème des Frottements." Annales de l'Institut Technique du Bâtiment et des Travaux Publics. Vol. 78. June 1954. p. 513. Serie: Beton Precontraint (19).

15. A paper on the tests performed with lubricated prestressing cables at Tancarville (France), in conjunction with the construction of Europe's longest suspension bridge, will be given at the May, 1958, Berlin Conference on Prestressed Concrete.

16. To be published elsewhere. Even when K and μ are both equal to zero, measured elongations are always larger than predicted elongations due to other considerations.

17. D. F. Billet and J. H. Appleton. "Flexural Strength of Prestressed Concrete Beams." Journal ACI. Vol. 25. June 1954.

18. D. F. Billet and J. H. Appleton. "Flexural Strength of Prestressed Concrete Beams." Journal ACI. Vol. 25. June 1954. p. 849.

19. D. F. Billet and J. H. Appleton. "Flexural Strength of Prestressed Concrete Beams." Journal ACI. Vol. 25. June 1954. p. 839. (The laboratory specimens were confined to a set with b/d in the neighborhood of 0.7, b itself kept nearly constant).

20. D. F. Billet and J. H. Appleton. "Flexural Strength of Prestressed Concrete Beams." Journal ACI. Vol. 25. June 1954. pp. 848, 849.

21. Jean Muller. "Quelques Aspects de la Resistance a la Rupture des Poutres Continues Precontraintes." Travaux. Juin 1957.

T. Y. LIN,[1] M. ASCE.--Being a member of the Committee, perhaps the writer is not in a position to praise its accomplishments. But he would like to express his opinion that the Committee certainly performed a great service to the profession. These Recommendations are urgently needed in this fast growing field of prestressed concrete.

The Recommendations covered pretty well the essential points required for the design and construction of simple prestressed concrete structures.

1. Prof. of Civ. Eng., Univ. of California, Berkeley, Calif.

It is obviously not possible that all Committee members could agree to the last detail. The writer, for one, has a number of dissentions. But he would like to limit this discussion to one basic disagreement concerning the approach to the design theory for prestressed concrete.

Article 201.3 of the Recommendations states:

"DESIGN THEORY—The elastic theory should be used at design loads with internal stresses limited to recommended values. The ultimate strength theory also should be applied to insure that ultimate capacity provides the recommended load factors."

The writer does not think this statement gives the best approach to design. First, correct elastic theory is not yet available, and probably never will be, for the design of certain important items such as shear and non-prestressed reinforcements in prestressed concrete. Secondly, the safety, economy, and behavior of structures will vary widely when designed according to a simple set of allowable stresses such as listed in the Recommendations. These facts were brought out by the writer in several papers.(1,2)

While it is true that such simple set of allowable stresses have been used for some time and have apparently yielded safe results, it must be realized that this was more a matter of coincidence than of rational justification. Most of these values were empirically employed by pioneers of prestressed concrete, who at that time did not have as much knowledge and data as we now have, or as we will have. We as engineers who endeavour to seek the truth and to apply the laws of nature should not blindly follow these empirical values.

As an example of the dangerous errors contained in these allowable stresses, let us consider the temporary stresses allowed in Article 207.3. Here tension in the concrete is limited to $3\sqrt{f'_{ci}}$ for single elements and zero for segmental elements. A recently completed investigation at the University of California(3) proved definitely that the strength and behavior of beams at transfer cannot be simply described by stresses but are dependent upon a number of factors, such as the shape of the section, the amount and location of prestress, etc.

Another pitfall in the elastic theory is discovered when applied to non-prestressed reinforcements intended to carry tensile stress. On account of shrinkage and creep in concrete, the non-prestressed reinforcements are actually in compression under the design loads. How can they be correctly designed by the elastic theory?

Better approach to the design theory has been recommended by other organizations. The Prestressed Concrete Institute Specifications state:

"Design shall be based on both the elastic and the ultimate theory. It must be ascertained that the resulting structure will perform properly under the service conditions and will possess strength to carry occasional overloads. Proper performance shall mean that camber, deflections, creep, shrinkage, and cracks are within control."

The Report of the Prestressed Concrete Committee of the Structural Engineers Association of Southern California included a similar statement but permits the design to be based on either the elastic or the ultimate theory, provided that proper behavior and sufficient ultimate strength can be assured. The reasoning there was that not both theories were available for all design problems, and many structures can be safely designed even if one theory is lacking.

To serve as guidance, both of the above two references recommend a set of allowable stresses. But they permit variations therefrom provided it is shown that the structure will behave properly under service conditions and will meet the requirements of overloads. They also caution that uncommon structures must be investigated for proper behavior and ultimate strength, even if all stresses are within the specified limits.

At this stage of our knowledge regarding prestressed concrete, we are not in a position to fix definite allowable values for all the stresses under all conditions. Arbitrary values as now recommended could not only end in uneconomical results, but could be misleading, and therefore dangerous. Furthermore, why should we try to freeze our progress at a beginner's level, when the realm of possibilities is just developing in front of us?

The writer strongly feels that the recommendations should aim at guiding the engineers in their thinking, helping them in their understanding, supplying them with facts, but not forcing them to blindly follow arbitrary rules of thumb. To this end, Article 201.3 should be revised along the directions set by the two other Reports mentioned previously. Several related clauses should also be clarified accordingly.

The engineering profession is perhaps in its renaissance, now that we engineers have finally awakened to a new approach in structural design: the understanding of the behavior and strength of structures. The Committee would take an important step forward if such an attitude is expressed in its Recommendations. Let us have enough courage to place the fundamentals of engineering design before the eyes of our fellow engineers, and not to mystify certain stress values which have no real meaning, nor to legalize certain rules of thumb which we know are only half correct.

REFERENCES

1. "Can We Design Prestressed Concrete by Allowable Stresses?," T. Y. Lin, Journal, Prestressed Concrete Institute, May, 1956.
2. "Load Factors for Prestressed Concrete Bridges," T. Y. Lin, Journal, Structural Division, ASCE. July, 1957.
3. "Flexural Strength of Prestressed Concrete Beams at Transfer," A. C. Scordelis, T. Y. Lin, and H. R. May, Proceedings, World Conference on Prestressed Concrete, 1957.

ANALYSIS OF BRACED FRAMES[a]

Discussion by George D. Siegfried

GEORGE D. SIEGFRIED,[1] J.M. ASCE.—Mr. Gerstle's paper presents a useful method of analysis for a very interesting class of building frames. A simpler method of computing the joint distribution factors is given below which may be found more attractive for application of the method to design problems.

Referring to the accompanying figure (which is similar to Figure 4 in the author's paper) the resisting forces of the members for an arbitrary rotation, θ, of the joint 0 are:

$$\begin{aligned} P_1 &= K_1\Delta_1 = K_1a_1\theta \\ P_2 &= K_2\Delta_2 = K_2a_2\theta \\ P_3 &= K_3\Delta_3 = K_3a_3\theta \end{aligned} \qquad (1)$$

Also, by summation of the moments of resisting forces and unbalanced forces about joint 0:

$$P_1a_1 + P_2a_2 + P_3a_3 = \sum Pa \qquad (2)$$

From equations (1) and (2):

$$P_1 = \frac{K_1a_1}{K_1a_1^2 + K_2a_2^2 + K_3a_3^2} \sum Pa$$

$$P_2 = \frac{K_2a_2}{K_1a_1^2 + K_2a_2^2 + K_3a_3^2} \sum Pa$$

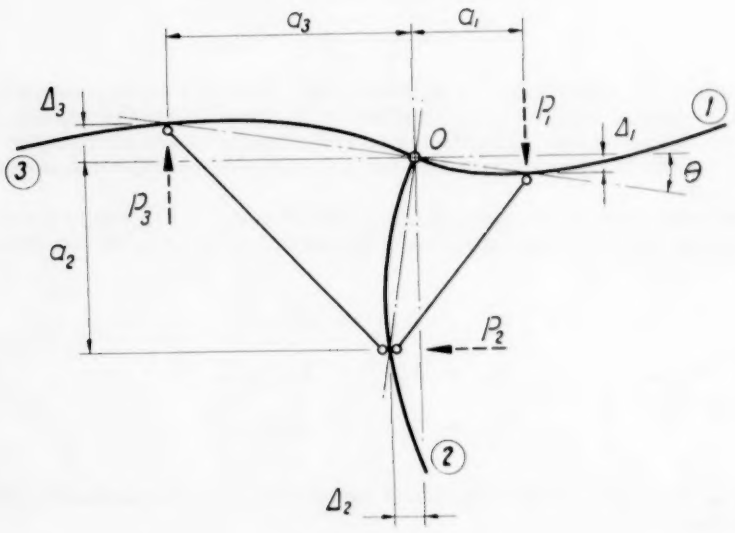
$$P_3 = \frac{K_3a_3}{K_1a_1^2 + K_2a_2^2 + K_3a_3^2} \sum Pa$$

The coefficients of ΣPa are the distribution factors for members 1, 2, and

a. Proc. Paper 1560, March, 1958, by Kurt H. Gerstle.

1. Civ. Engr., Smith and Moorehead, Structural Engrs., San Francisco, Calif.

3 respectively. Note that the factors are coefficients of the unbalanced moment about the joint 0 which corresponds to the method of moment distribution for rigid jointed frames. Three factors are necessary in the general case for a joint having two braces rather than nine as given by the author. Stiffness factors, carry-over factors, and fixed support forces are of course the same as given in the author's paper.



LATERAL BRACING OF COLUMNS AND BEAMS[a]

Discussion by William Zuk

WILLIAM ZUK.[1]—As is characteristic of Professor Winter, he has again very aptly reduced and synthesized an involved problem of many variables to one of familiar principles easy to perceive. Since he has clearly defined his objectives and limitations, there can be no grounds for objection to his statements of simplification. However, certain comments may be made in regard to beam bracing for the benefit of practicing engineers who are not familiar with the involved theory of lateral buckling but who are concerned with safe, economical bracing. These comments are thus not intended for Dr. Winter, who is quite aware of these facts already. The section on column bracing is neatly treated, so that all comments in this discussion apply only to beams.

Professor Winter's approximate analysis of beam bracing conservatively assumes the compression flange to stand alone in the lateral direction and thus to act as a column. Several limitations of this assumption suggest themselves where the approximate theory would be inadequate. The following are several such situations:

a) Beams whose bracing is not in the plane of the compression flange as in the case of tie rods and knee braces in a through plate girder bridge.
b) Beams with an initial torsional twist, with or without a bow in the compression flange.
c) Beams in which the compression flange force changes magnitude between points of bracing, particularly for beams whose flange stress changes from compression to tension as beams with an overhang.

Discussing each of these situations separately, case (a) is discussed by the writer in the ASCE paper 1032 "Lateral Bracing Forces on Beams and Columns" where the required force on I beams is presented for bracing positioned at any level on the beam, taking into account the torsional-flexure behavior. The results show that the full lateral buckling value may be obtained for the beam if the brace is anywhere in the compression region, from the neutral axis to the compression flange. However, the required bracing force becomes less as the position of the brace approaches the compression flange. Even for a brace at the neutral axis of flexure, the bracing force is still only a few percent of the compression flange force; substantiating Professor Winter's views that the required lateral bracing is a relatively small value. It may also be noted that "partial bracing" is attained even for beams braced at the tension flange. Normally beams braced at the tension flange are considered unsupported and designed accordingly; however, the critical lateral

a. Proc. Paper 1561, March, 1958, by George Winter.
1. Assoc. Prof. of Civ. Eng., Univ. of Virginia, Charlottesville, Va.

buckling moment actually lies between the fully braced and unbraced value. The exact value of the buckling moment and bracing force for this case is also presented in paper 1032.

Situation (b) where the beam is initially twisted, even though no compression flange bowing is present will also generate a lateral bracing force. This case, too, is presented in theoretical form in paper 1032, but is too complex for general design use. It is quite possible however, that an equivalent bowing factor could also include the effects of twist, so that Professor Winter's simplified design criteria could well serve the design engineer.

Situation (c) is one which falls outside the scope of any simple approximate analysis as proposed by Professor Winter. A more rigorous method taking into account torsion is called for to solve this situation with any degree of reliability. However, it is believed that an order of magnitude study may be made for this case using Professor Winter's procedure if the points of contraflexure are assumed as ficticious bracing points, with the buckling value based on this distance between the real or ficticious bracing points.

From these few fragmentary comments, it is obvious that quite a good deal more research is needed to exactly understand the nature of bracing forces on beams, including trusses of straight and curved compression chords. No conclusive test evidence is yet available for beam bracing, although some special cases of theoretical solutions are known. It is the writer's opinion that column bracing behavior is now on fairly sure ground and that future efforts should be concentrated on the more complex case of beam bracing behavior. Professor Winter's contribution to beam bracing behavior stands analogous to the lateral beam buckling criteria of $\frac{L}{b} < 40$ of former years based on a freed compression flange in relation to the present more exact criterial of $\frac{Ld}{bt} < 600$ based on torsion-flexure. (Reference to these relations is made from the standard AISC specifications for I beams.) The contribution is a valuable one but perhaps not a final one.

THE DESIGN OF THE MAIN TOWERS OF THE MACKINAC BRIDGE[a]

Corrections by Kuang-Han Chu

CORRECTIONS.—In the last line on page 1565-5, the last sentence

" . . . of approximately 29 Kips ($0.44F_1$ = 0.44 x 66) a negligible amount with respect to total H."

should be corrected as follows:

" . . . of approximately 29 Kips. (Of the 3.08 ft total deflection, 2.64 ft is attributed to bending due to vertical load including tower weight, and 0.44 ft to the difference in H. Hence $\delta H = 0.44F_1 = 0.44 \times 66 = 29$ Kips.) This is a negligible amount with respect to total H."

The caption for Fig. 12 should be changed to:

"View of the Completed North Tower in December, 1955"

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a. Proc. Paper 1565, March, 1958, by Kuang-Han Chu.

PROCEEDINGS PAPERS

The technical papers published in the past year are identified by number below. Technical-division sponsorship is indicated by an abbreviation at the end of each Paper Number, the symbols referring to: Air Transport (AT), City Planning (CP), Construction (CO), Engineering Mechanics (EM), Highway (HW), Hydraulics (HY), Irrigation and Drainage (IR), Pipeline (PL), Power (PO), Sanitary Engineering (SA), Soil Mechanics and Foundations (SM), Structural (ST), Surveying and Mapping (SU), and Waterways and Harbors (WW), divisions. Papers sponsored by the Board of Direction are identified by the symbols (BD). For titles and order coupons, refer to the appropriate issue of "Civil Engineering." Beginning with Volume 82 (January 1956) papers were published in Journals of the various Technical Divisions. To locate papers in the Journals, the symbols after the paper numbers are followed by a numeral designating the issue of a particular Journal in which the paper appeared. For example, Paper 1449 is identified as 1449 (HY 6) which indicates that the paper is contained in the sixth issue of the Journal of the Hydraulics Division during 1957.

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JULY: 1289(SM3), 1290(EM3), 1291(EM3), 1292(EM3), 1293(EM3), 1294(HW3), 1295(HW3), 1296(HW3), 1297 (HW3), 1298(HW3), 1299(SM3), 1300(SM3), 1301(SM3), 1302(ST4), 1303(ST4), 1304(ST4), 1305(SU1), 1306 (SU1), 1307(SU1), 1308(ST4), 1309(SM3), 1310(SU1)[c], 1311(EM3)[c], 1312(ST4), 1313(ST4), 1314(ST4), 1315 (ST4), 1316(ST4), 1317(ST4), 1318(ST4), 1319(SM3)[c], 1320(ST4), 1321(ST4), 1322(EM3), 1323(AT1), 1324 (AT1), 1325(AT1), 1326(AT1), 1327(AT1), 1328(AT1)[c], 1329(ST4)[c].

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OCTOBER: 1387(CP2), 1388(CP2), 1389(EM4), 1390(EM4),1391(HY5), 1392(HY5), 1393(HY5), 1394(HY5). 1395 (HY5), 1396(PO5), 1397(PO5), 1398(PO5), 1399(EM4), 1400(SA5), 1401(HY5), 1402(HY5), 1403(HY5), 1404 (HY5), 1405(HY5), 1406(HY5), 1407(SA5), 1408(SA5), 1409(SA5), 1410(SA5), 1411(SA5), 1412(EM4), 1413 (EM4), 1414(PO5), 1415(EM4)[c], 1416(PO5)[c], 1417(HY5)[c], 1418(EM4), 1419(PO5), 1420(PO5), 1421(PO5), 1422(SA5)[c], 1423(SA5), 1424(EM4), 1425(CP2).

NOVEMBER: 1426(SM4), 1427(SM4), 1428(SM4), 1429(SM4), 1430(SM4)[c]. 1431(ST6), 1432(ST6), 1433(ST6), 1434(ST6), 1435(ST6), 1436(ST6), 1437(ST6), 1438(SM4), 1439(SM4), 1440(ST6), 1441(ST6), 1442(ST6)[c], 1443(SU2), 1444(SU2), 1445(SU2), 1446(SU2), 1447(SU2), 1448(SU2)[c].

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c. Discussion of several papers, grouped by divisions.

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